

# Super Yang-Mills Theory in 10+2 Dimensions, The 2T-physics Source for $\mathcal{N}=4$ SYM and M(atrix) Theory\*

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## Abstract

In this paper we construct super Yang-Mills theory in 10+2 dimensions ( $\text{SYM}_{10+2}^1$ ), a number of dimensions that was not reached before in a unitary supersymmetric field theory, and show that this is the 2T-physics source of some cherished lower dimensional field theories. The much studied conformally exact  $\mathcal{N}=4$  SuperYang-Mills field theory in 3+1 dimensions ( $\text{SYM}_{3+1}^4$ ) is known to be a compactified version of  $\mathcal{N}=1$  SYM in 9+1 dimensions ( $\text{SYM}_{9+1}^1$ ), while M(atrix) theory is obtained by compactifications of the 9+1 theory to 0 dimensions (also 0+1 and others). We show that there is a deeper origin of these theories in two higher dimensions as they emerge from the new  $\text{SYM}_{10+2}^1$  theory with two times. Pursuing various alternatives of gauge choices, solving kinematic equations and/or dimensional reductions of the 10+2 theory, we suggest a web of connections that include those mentioned above and a host of new theories that relate 2T-physics and 1T-physics field theories, all of which have the 10+2 theory as the parent. In addition to establishing the higher spacetime underpinnings of these theories, a side benefit could be that in principle our approach can be used to develop new computational techniques.

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## I. THE ACTION, AND SUMMARY OF RESULTS

The rules of 2T field theory in flat space [1]-[5] and in curved space [6]-[10] in  $d + 2$  dimensions are well established [9]. They are derived from an underlying  $\text{Sp}(2, R)$  gauge symmetry principle in phase space at the worldline level, which in turn leads to a *ghost free unitary 2T field theory* thanks to new gauge symmetries and kinematic constraints that follow from the 2T field theory action. These gauge symmetry principles provide the fundamental answer to the question of how to construct a physical theory in a spacetime with two timelike dimensions and still avoid unitarity and causality problems. 2T field theory constructed with these rules is compatible with conventional 1T field theory, but beyond this consistency, 2T-physics makes predictions that are missed in 1T physics systematically. These include dualities and hidden symmetries [4][5] and restrictions on the interactions of scalar fields in usual relativistic 1T field theory [1][7][8][10]. The new predictions, in the contexts of classical mechanics, quantum mechanics or field theory, are all consistent with known phenomenology at all scales of physics explored so far [9].

Following these rules the Lagrangian for the vector supermultiplet  $(A_M^a, \lambda_A^a)$  and its coupling to gravity fields  $(G_{MN}, \Omega, W)$  in special  $d + 2 = 5, 6, 8, 12$  dimensions is constructed uniquely as follows

$$S = S_{SYM} + S_{Gravity} + \dots \quad (1.1)$$

where the part that concerns this paper is

$$S_{SYM} = K \int d^{d+2} X \sqrt{-G} \delta(W(X)) \left\{ -\frac{1}{4g_{YM}^2} \Omega^{2\frac{d-4}{d-2}} F_{MN}^a F_a^{MN} + \frac{i}{2} \left[ \bar{\lambda}^a V \bar{D} \lambda^a + \bar{\lambda}^a \overleftarrow{D} V \lambda^a \right] \right\}. \quad (1.2)$$

Note the unusual but important factor  $\delta(W(X))$  in the volume element which is essential in 2T field theory, where the  $W(X)$  field, along with the dilaton field  $\Omega(X)$  are members of the “gravity triplet”  $(G_{MN}, \Omega, W)$ . In  $L_{SYM}$ , the Yang-Mills field  $A_M^a(X)$  is a vector in  $d + 2$  dimensions  $X^M$  with  $M = 0', 1', 0, 1, \dots, (d - 1)$ , with two timelike components,  $0, 0'$ , while  $\lambda_A^a(X)$  is a Weyl or Majorana spinor of  $\text{SO}(d, 2)$  (with 32 real components for  $d + 2 = 12$  labelled by  $A = 1, 2, \dots, 32$ ). Both  $A_M^a, \lambda_A^a$  are in the adjoint representation of the Yang-Mills gauge group  $G$  with  $a = 1, 2, \dots, \dim(adj)$ . The scalar fields  $\Omega$  and  $W$ , together with the metric  $G_{MN}$ , are necessary to build up the action  $S_{Gravity}$  for 2T-Gravity

as recently constructed [6] and analyzed in great detail [7][8],

$$S_{Gravity} = K \int d^{d+2} X \sqrt{G} \left[ \begin{aligned} &\delta(W) \{a_d \Omega^2 R(G) + \frac{1}{2} \partial \Omega \cdot \partial \Omega - V(\Omega)\} \\ &+ \delta'(W) \{a_d \Omega^2 (4 - \nabla^2 W) + a_d \partial W \cdot \partial \Omega^2\} \end{aligned} \right] \quad (1.3)$$

Our spinor conventions and gamma matrices  $\Gamma^i, \bar{\Gamma}^i$  for  $SO(10, 2)$  are given in footnote (9) and in great detail in the appendix of ref.[2]. The symbols  $V \equiv \Gamma^M V_M = \Gamma^i V_i$  and  $\bar{V} \equiv \bar{\Gamma}^M V_M = \bar{\Gamma}^i V_i$ , with gamma matrices  $\Gamma^M = \Gamma^i E_i^M$  that appear in  $L_{SYM}$ , contains the fields  $V_M(X) \equiv \frac{1}{2} \partial_M W$  and the vielbein  $E_M^i$  explained in Eq.(1.5) below.

The supersymmetric completion of  $S_{Gravity}$  to supergravity symbolized by “ $+\dots$ ” in Eq.(1.1) has been obtained in 4+2 dimensions [11], but it remains incomplete in 10+2 dimensions at this stage. Therefore, in this paper the fields  $\Omega, W, G_{MN}$  will be treated as if they are non-dynamical backgrounds for the purpose of supersymmetry transformations. So,  $W, \Omega, G_{MN}$  will not transform under SUSY. Except for *kinematical* equations (not dynamical ones, see below) of the background fields  $\Omega, W, G_{MN}$  given in Eq.(1.5), that follow from varying  $S_{Gravity}$  [6][7], without involving  $S_{SYM}$ , or the missing terms “ $\dots$ ”, the gravitational sector ( $S_{Gravity} + \dots$ ) will not play a further role in determining the supersymmetry or other structural properties of  $S_{SYM}$ . This is sufficient to construct and interpret SYM in  $d+2 = 5, 6, 8, 12$ .

In 2T field theory, the gauge field  $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + f^{abc} A_M^b A_N^c$  must couple to  $\Omega$  and  $G_{MN}$  in the action in the form  $-\frac{1}{4g^2} \Omega^2 \frac{d-4}{d-2} F_{MN}^a F_{PQ}^a G^{MP} G^{NQ}$  in  $d+2$  dimensions, but note that the dilaton factor disappears in 4+2 dimensions. The covariant derivative of the spinor  $\bar{D}\lambda_A^a$  contains the  $SO(d, 2)$  spin connection  $\omega_M^{ij}$  and the inverse vielbein  $E_k^M$ , in addition to the Yang-Mills field  $A_M^a$ . Note also the notation  $\bar{\lambda}^a \overleftarrow{D} \equiv \overleftarrow{D}_M \bar{\lambda}^a \Gamma^M$ .

$$\bar{D}\lambda^a \equiv \bar{\Gamma}^M D_M \lambda^a = \bar{\Gamma}^k E_k^M \left( \partial_M \lambda^a + \frac{1}{4} \omega_M^{ij} \Gamma_{ij} \lambda^a + f^{abc} A_M^b \lambda^c \right). \quad (1.4)$$

The variation of the action with respect to each field produces terms proportional to  $\delta(W), \delta'(W)$  and  $\delta''(W)$ . Each coefficient must vanish since these are linearly independent distributions. The terms proportional to  $\delta'(W)$  and  $\delta''(W)$  are the “kinematic” equations while the terms proportional to  $\delta(W)$  are the “dynamical” equations. The dynamical equations for each field contain interactions, but the kinematical ones do not (except for those that enter through gauge invariant derivatives, but those interactions vanish in special gauge choices). So, in addition to the usual geometric relations (as found in standard general relativity textbooks) among the vielbein  $E_M^i$ , metric  $G_{MN} = E_M^i E_N^j \eta_{ij}$ , spin connection  $\omega_M^{ij}$ ,

and affine connection  $\Gamma_{MN}^P$  (not to be confused with gamma matrices), the following *additional kinematic equations* among geometrical quantities in 2T gravity in  $d + 2$  dimensions must also be imposed on the gravitational fields  $W, \Omega, G_{MN}, E_M^i, \omega_M^{ij}, \Gamma_{MN}^P$ . It is important to emphasize that these “kinematic” equations follow from varying the action for  $S_{Gravity}$ ; they are not imposed from outside as additional constraints.

$$\begin{aligned}
V_M &= \frac{1}{2} \partial_M W, \quad V^M = G^{MN} V_N, \quad V^i = V^M E_M^i, \\
W &= V^i V_i = G^{MN} V_M V_N = \frac{1}{2} V^M \partial_M W, \\
G_{MN} &= \nabla_M V_N = \frac{1}{2} (\partial_M \partial_N W - \Gamma_{MN}^P \partial_P W), \\
E_M^i &= D_M V^i = \partial_M V^i + \omega_M^{ij} V_j, \\
(V^M \partial_M + \frac{d-2}{2}) \Omega &= 0.
\end{aligned} \tag{1.5}$$

These equations can also be derived directly from the  $\text{Sp}(2, R)$  gauge symmetry principle that underlies 2T-physics at the worldline level in a curved background that includes  $G_{MN}$  [6]. The significance of these kinematical equations is to restrict the degrees of freedom to gauge invariant sectors of the underlying  $\text{Sp}(2, R)$  gauge symmetry in curved backgrounds [6][7][10]. Through these equations, the scalar field  $W$  determines some of the properties of geometrical quantities such as  $G_{MN}, E_M^i$ , etc. Geometrically, these are *homothety conditions* on the metric  $G_{MN}$  and other fields [6][7][10].

These equations are solved by flat spacetime in  $d + 2$  dimensions as well as by the most general curved spacetime in  $d$  dimensions (less one time and one space dimension) as embedded in  $d + 2$  dimensions<sup>1</sup>. In the general solution there are no Kaluza-Klein type degrees of freedom that connect the “shadow” in  $d$  dimensions and the “substance” in  $d + 2$  dimensions. There are prolongations of the shadow [7] that extend into  $d + 2$  dimensions, but they are constructed from the degrees of freedom of the shadow within  $d$  dimensions, and these prolongations do not play any role in determining the 1T-physics observed within the

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<sup>1</sup> Flat space in  $d + 2$  dimensions obeys Eq.(1.5) with  $W_{flat} = X \cdot X = \eta_{MN} X^M X^N$ ,  $V_M^{flat} = X_M$ ,  $G_{MN}^{flat} = \eta_{MN}$ ,  $(\Gamma_{MN}^P)_{flat} = 0 = (\omega_M^{ij})_{flat}$  and  $\Omega_{flat} = (c \cdot X)^{1-d/2}$  with a constant  $c_M$ . A curved metric that satisfies Eq.(1.5) can be taken in the form  $G_{MN} = \eta_{MN} + h_{MN}(X)$ , still with  $W = \eta_{MN} X^M X^N$ ,  $V^M = X^M$ ,  $V_M = \eta_{MN} X^N$ , but with  $X^M h_{MN} = 0$ ,  $X \cdot \partial h_{MN} = 0$ . Other forms of solutions of Eq.(1.5) in curved space, that are more convenient to describe the conformal shadow, are found in [6][7]; see also the text and appendix-A in this paper. The solution for all such  $G_{MN}$  corresponds to the most general unrestricted background metric  $g_{\mu\nu}(x)$  in  $d$  dimensions  $x^\mu$  [12] plus “prolongations” in the extra dimensions [7]. Depending on the shadow (see footnote 2), the prolongations are determined by  $g_{\mu\nu}(x)$  or are gauge freedom; they are not dynamical Kaluza-Klein modes.

shadow.

In this paper we will establish the properties of this theory as summarized in Fig.1. In section II we will discuss the central box at the top of Fig.1 for  $\text{SYM}_{d+2}^1$  for  $d+2 = 12, 8, 6, 5$  and argue that SUSY holds thanks to the following two essential properties

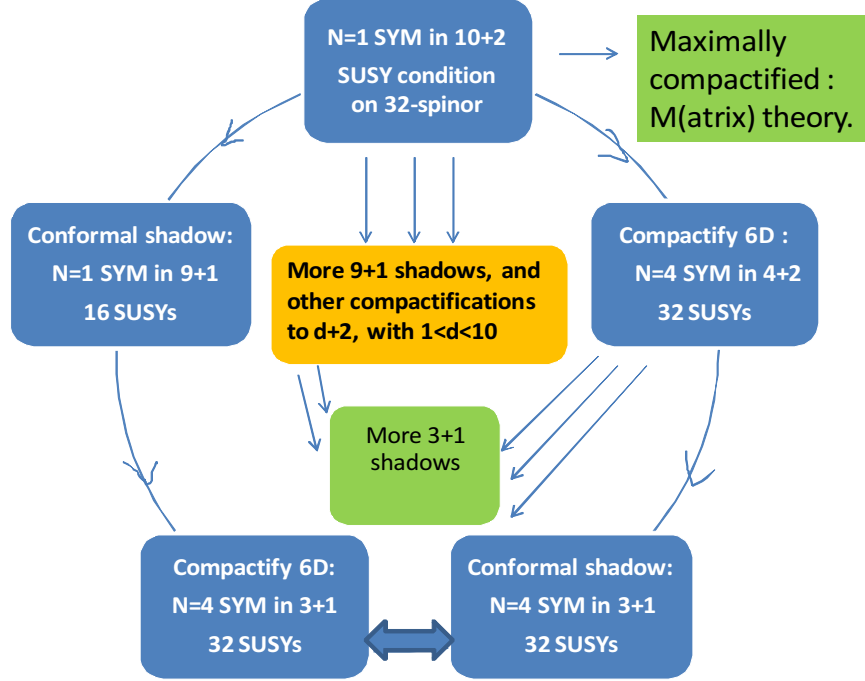


Fig.1 -  $\text{SYM}_{10+2}^1$  is the parent of  $\text{SYM}_{9+1}^1$ ,  $\text{SYM}_{4+2}^4$ ,  $\text{SYM}_{3+1}^4$ , and M(atrix) theories.

- 1- The first ingredient is the following special identity for the gamma matrices of  $\text{SO}(d, 2)$ ,  $\Gamma^{ij} = \frac{1}{2} (\Gamma^i \Gamma^j - \Gamma^j \Gamma^i)$ ,

$$(\Gamma^{ik})_{(AB)} (\Gamma_k^j)_{(C)D} + (\Gamma^{jk})_{(AB)} (\Gamma_k^i)_{(C)D} = \frac{2\eta^{ij}}{d+2} (\Gamma^{kl})_{(AB)} (\Gamma_{lk})_{(C)D}. \quad (1.6)$$

We derived this property and showed that it is satisfied only for  $d+2 = 12, 8, 6, 5$ . Here the  $\text{SO}(d, 2)$  spinor indices are symmetrized as implied by the parenthesis  $(ABC)$ .

- 2- In addition, the local SUSY parameter  $\varepsilon_A(X)$  must obey the following differential condition in the presence of the curved spacetime backgrounds  $G_{MN}, \Omega, W$  consistently with Eq.(1.5)

$$\left\{ -\frac{d-4}{d-2} (\bar{\Gamma}^{PQN} \Gamma^M \varepsilon)_A V_N \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{PQN} D_M \varepsilon)_A V_N = V^P U_A^Q - V^Q U_A^P \right\}_{W=0} \quad (1.7)$$

The  $U_A^Q(X)$  in Eq.(1.7) is an arbitrary vector-spinor. Solutions of this equation will be discussed at the end of section II and in Appendix (B). We emphasize that the SUSY condition (1.7) arises because the background fields do not transform under SUSY. In dynamical 2T supergravity in  $d+2$  dimensions [11], where  $G_{MN}, \Omega, W$  and the gravitino  $\Psi_{MA}$  also undergo SUSY transformations, the transformation of the gravitino field,  $\delta_\epsilon \Psi_{MA} = D_M \epsilon_A + \dots$ , will cancel at least the  $D_M \epsilon$  part of this expression thus removing or altering this condition on  $\epsilon_A(X)$ .

In section IIIA we will outline the derivation of  $\text{SYM}_{9+1}^1$  as the “conformal shadow” of  $\text{SYM}_{10+2}^1$  taken in a *flat background*. The conformal shadow<sup>2</sup> is arrived at as a combination of a special gauge choice of 2T gauge symmetries and the solution of kinematic constraints on  $A_M^a, \lambda_A^a$  derived from  $L_{SYM}$ . It is well known that  $\text{SYM}_{9+1}^1$  has only 16 supersymmetries while its compactification to  $\text{SYM}_{3+1}^4$  depicted in Fig.1 results in the intensely studied  $\mathcal{N} = 4$  SYM theory in  $3+1$  dimensions with 32 supersymmetries within the supergroup  $\text{SU}(2, 2|4)$ .

In section IIIB we will obtain  $\text{SYM}_{4+2}^4$  as a straightforward compactification of the Lagrangian  $L_{SYM}$  into a 2T field theory in flat  $4+2$  dimensions with 32 supersymmetries. The unique  $\text{SYM}_{4+2}^4$  theory was previously constructed by us by direct 2T SUSY methods in  $4+2$  dimensions [3]. We had previously argued that the “conformal shadow” of the unique  $\text{SYM}_{4+2}^4$  is the  $\mathcal{N} = 4$  super Yang-Mills theory  $\text{SYM}_{3+1}^4$ . So, the left and right sides in Fig.1 arrive at the same special  $\text{SYM}_{3+1}^4$  with 32 supersymmetries via different routes that display the consistency and some of the properties of the parent  $\text{SYM}_{10+2}^1$  theory. We expect that the parent  $\text{SYM}_{10+2}^1$  theory, and its compactification to  $\text{SYM}_{4+2}^4$ , together with the various shadows that are related by dualities [4][5], would add new tools and shed new light on the intensely studied  $\text{SYM}_{3+1}^4$  theory.

In section IV we will derive the 2T version of M(atrix) theory (top right in Fig.1) by

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<sup>2</sup> Nontrivial examples of 1T shadows from 2T-physics has been given in classical or quantum mechanics and in field theory. For the simplest flat  $d+2$  dimensional background described in footnote (1) see the figures in [9] and the corresponding formulas for the shadows summarized in tables I, II and III in [4], which includes examples of shadows in field theory (see also [5]). The conformal shadow is the one most familiar to particle physicists. Therefore it has featured as an explicit example in many old [13]-[22] and recent discussions [23] in addition to discussions in many papers by the current authors [9], to help absorb some of the 1T physical content in 2T-physics. The richness of the predictions of 2T-physics, which is missed in the conventional formulation of 1T-physics, is in the presence of the many other shadows such as those summarized in [9][24].

dimensionally reducing 9 or 9+1 dimensions, leaving behind a 2T M(atrrix) theory with either 1+2 or 1+1 dimensions with certain gauge symmetries. The conformal shadows of this 2T M(atrrix) theory in 1+1 or 1+2 dimensions yields the familiar versions of 1T M(atrrix) theory that describe (-1)-branes, 0-branes, or more generally p-branes [25]-[31].

There are many other routes of deriving supersymmetric theories from  $\text{SYM}_{10+2}^1$  either by exploring the many shadows of 2T-physics other than the conformal shadow or by considering other compactifications, as well as a variety of backgrounds  $G_{MN}, \Omega, W$ . These are indicated schematically in Fig.1. We will make only brief comments on these possibilities.

## II. SUSY CONDITION

The SUSY transformation of the dynamical fields  $A_M^a, \lambda_A^a$  is similar to the one we discussed previously in 4 +2 dimensions [2][3] but here it is modified for  $d + 2 = 5, 6, 8, 12$  dimensions and the presence of the background fields,  $G_{MN}, \Omega, W, \omega_M^{ij}, E_M^i$ , which were absent in [2][3]:

$$\delta_\varepsilon \lambda_A^a = \frac{i}{g_{YM}} \Omega^{\frac{d-4}{d-2}} F_{MN}^a (\Gamma^{MN} \varepsilon)_A, \quad \delta_\varepsilon A_M^a = \Omega^{-\frac{d-4}{d-2}} [-2\bar{\varepsilon} \Gamma_M \bar{V} \lambda^a + W \bar{\varepsilon} \Gamma_{MN} D^N \lambda^a] + h.c.. \quad (2.1)$$

It takes some effort to verify that the action (1.1) is invariant  $\delta_\varepsilon S = 0$  under (2.1,2.3), *without varying the background fields*. After taking into account the kinematic properties of the curved background in Eq.(1.5), which is discussed in detail in Sec.IIIB of [7], one finds that the algebraic manipulations to verify SUSY are completely parallel to those in flat 4 + 2 dimensions given in [2], and the proof proceeds by formally replacing derivatives by covariant derivatives, etc., in the presence of the backgrounds. So, we will only state that indeed we find  $\delta_\varepsilon S = 0$  by following the steps of the computation in [2]. The crucial equations (1.6,1.7) are the only new ingredients necessary to show the symmetry of the action in  $d + 2$  dimensions, with  $d + 2 = 12, 8, 6, 5$ . In particular, the SUSY condition (1.7) is new (but trivially satisfied for  $d + 2 = 6$ ).

An alternative proof of supersymmetry is to show that there is a conserved SUSY current,  $0 = \partial_M (\bar{\varepsilon}^A J_A^M) = (D_M \bar{\varepsilon}^A) J_A^M + \bar{\varepsilon}^A D_M J_A^M$ , where  $D_M$  includes the background spin connection. The current derived by using Eqs.(2.1) and Noether's theorem is

$$\bar{\varepsilon} J^M = \delta(W) \sqrt{G} \Omega^{\frac{d-4}{d-2}} F_{PQ}^a V_N \bar{\varepsilon} (\Gamma^{PQN} \bar{\Gamma}^M) \lambda^a. \quad (2.2)$$

To show the conservation,  $\partial_M (\bar{\varepsilon} J^M) = 0$ , we must use the equations of motion derived from



the action. The “kinematic equations”, namely those that come from terms proportional to  $\delta'(W)$  in the variation of the action are<sup>3</sup>

$$V^M F_{MN}^a = 0, \quad \left( V \cdot D + \frac{d}{2} \right) \lambda_A^a = 0, \quad (2.4)$$

in addition to those listed in Eqs.(1.5,2.3). The “dynamical equations”, namely those that come from terms proportional to  $\delta(W)$  in the variation of the action, are<sup>4</sup>

$$(V \bar{D} \lambda^a)_A = 0, \quad \hat{D}_N \left( \Omega^{\frac{2(d-4)}{d-2}} F_a^{NM} \right) = f_{abc} \left( \bar{\lambda}^b \Gamma^{MN} \lambda^c \right) V_N. \quad (2.5)$$

These are required to be satisfied only on the shell  $W = 0$ .

Additional properties of this current<sup>5</sup> include that it is orthogonal to  $V_M$ , namely  $\bar{\varepsilon} J^M V_M = 0$ , proven by using the kinematic equations  $W = V \cdot V = 0$  and  $V^M F_{MN}^a = 0$  in (1.5,2.4) and applying  $W \delta(W) = 0$ . It can also be verified that this current is invariant under the following local symmetries shared by the action (see also footnote (8)) : (1) Under the 2T gauge transformation of the gauge field [1],  $\delta_\Lambda A_M^a = W s_M^a(X)$ , the totally antisymmetric form  $\delta(W) F_{[PQ]} V_N$  that occurs in the current is invariant, and (2) under the 2T gauge transformation of the gaugino  $\delta_\kappa \lambda_A^a = (V \kappa_1^a)_A + W \kappa_{2A}^a$  [1] with local fermionic parameters  $\kappa_{1A}^a(X), \kappa_{2A}^a(X)$ , the expression for  $\delta_\kappa(\bar{\varepsilon} J^M)$  vanishes modulo the irrelevant

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<sup>3</sup> In proving the conservation of the current we must also include a kinematic condition on the SUSY parameter

$$V \cdot D \varepsilon_A \equiv V^M \left( \partial_M \varepsilon_A + \frac{1}{4} \omega_M^{ij} (\Gamma_{ij} \varepsilon)_A \right) = 0. \quad (2.3)$$

This is required since in this computation all fields are on shell constrained by kinematic equations (as a result of equations of motion), whose significance is the imposition of  $\text{Sp}(2, R)$  gauge invariance.

<sup>4</sup> Identifying the dynamical/kinematical equations from a variation of the action that has the form  $\delta S \sim \int \delta \Phi [\alpha(X) \delta(W) + \beta(X) \delta'(W)] = 0$  requires also a discussion of gauge symmetry. For a recent discussion see section-V in [7] for how a gauge is chosen to arrive at the kinematic equation  $\beta = 0$  at all  $W(X)$ , and the dynamical equation  $\alpha = 0$  at  $W(X) = 0$ , and how this relates to an underlying  $\text{Sp}(2, R)$  symmetry.

<sup>5</sup> This current can be modified by additional inessential terms  $\Delta J_A^M$  that are automatically conserved  $\partial_M (\bar{\varepsilon} \Delta J^M) = 0$  on their own, independent of dynamics. Such terms, that are analogous to the automatically conserved terms in the “new improved” energy momentum tensor, have the forms  $\Delta J_A^M = \delta(W) \sqrt{G} V^M \xi_A$  or  $\Delta J_A^M = \delta'(W) \sqrt{G} V^M \tilde{\xi}_A$ , where the spinors  $\xi_A, \tilde{\xi}_A$  may be functions of the fields and must satisfy homogeneity conditions  $(V \cdot D + d) \xi_A = 0$  and  $(V \cdot D + d - 2) \tilde{\xi}_A = 0$  that follow only from the kinematical equations (1.5,2.3,2.4) for all fields including the backgrounds. An example is  $\Delta J_A^M = \delta(W) \sqrt{G} V^M \Omega^{\frac{d-4}{d-2}} F_{PQ}^a (\Gamma^{PQ} \lambda^a)_A$ . The automatic conservation is verified by noting some simple kinematic relations, such as  $\partial_M (\sqrt{G} V^M) = \sqrt{G} \nabla_M V^M = \sqrt{G} \delta_M^M = (d+2) \sqrt{G}$  and  $V \cdot \partial \delta(W) = \delta'(W) V \cdot \partial W = 2W \delta'(W) = -2\delta(W)$ , where  $V \cdot \partial W = 2W$  was used (Eq.1.5) and similarly  $V \cdot \partial \delta'(W) = -4\delta'(W)$ . Then with only the kinematics one verifies  $\partial_M (\bar{\varepsilon} \Delta J^M) = 0$ , *independent of the dynamical equations* (2.5).

types of terms described in footnote (5), or contains terms proportional to the kinematic equations (1.5,2.4) which also vanish. So, although the fermionic 2T gauge transformation of the current  $\delta_\kappa(\bar{\varepsilon}J^M)$  is not strictly zero, it may be ignored in the  $\text{Sp}(2, R)$  gauge invariant sector, since it vanishes when only the kinematic equations are put on shell, while the dynamic equations (2.5) are not imposed.

We emphasize the following crucial points in proving the conservation of the current  $\partial_M(\bar{\varepsilon}^A J_A^M) = 0$ . After using both the kinematic and dynamical equations of motion, the divergence of the current can be brought to the form

$$\partial_M(\bar{\varepsilon}J^M) = \sqrt{G}\delta(W) \left\{ 2\Omega^{-\frac{d-4}{d-2}} f_{abc} V_N V^P (\bar{\varepsilon}\Gamma^{QN}\lambda^a) (\bar{\lambda}^b \Gamma_{QP}\lambda^c) \right. \\ \left. + F_{PQ} V_N \left[ -\partial_M \Omega^{\frac{d-4}{d-2}} (\bar{\varepsilon}\Gamma^M \bar{\Gamma}^{PQN}\lambda) \right. \right. \\ \left. \left. + (D_M \bar{\varepsilon}) \Gamma^{PQN} \bar{\Gamma}^M \lambda \right] \right\}. \quad (2.6)$$

Now we use the special gamma matrix identity (1.6) in  $d+2$  dimensions (holds only for  $d+2 = 12, 8, 6, 5$ ) to show that the first term in (2.6) vanishes

$$f_{abc} V_N V^P (\bar{\varepsilon}\Gamma^{QN}\lambda^a) (\bar{\lambda}^b \Gamma_{QP}\lambda^c) \delta(W) = \frac{2}{d+2} f_{abc} (\bar{\lambda}^b \Gamma_{kl}\lambda^a) (\bar{\varepsilon}\Gamma^{kl}\lambda^c) W \delta(W) = 0. \quad (2.7)$$

The gamma matrix identity (1.6) produces the second form in (2.7), but this identity alone is not sufficient to eliminate the first term in (2.6); we also need  $W\delta(W) = 0$  as in the last step of (2.7). The remaining expression in (2.6) is in general non-vanishing. However, if the SUSY parameter  $\varepsilon^A(X)$  satisfies the condition (1.7) then this also vanishes after using the kinematic equations,  $V^M F_{MN}^a = 0$ ,  $W = V \cdot V$  and  $W\delta(W) = 0$ , for any  $U_A^P(X)$  in (1.7).

The discussion above provides an outline of the proof that  $\text{SYM}_{10+2}^1$  is indeed supersymmetric when  $\varepsilon_A(X)$  satisfies the SUSY condition (1.7). Now we want to show that there are solutions for  $\varepsilon_A$  that satisfy this condition. All solutions of Eq.(1.7) are obtained in Appendix (B) by concentrating on the conformal shadow. Below we display a specialized subclass of simpler looking solutions that share some of the main features of the general solution.

The simple class that obviously solves Eq.(1.7) is defined by imposing stronger conditions on  $\varepsilon_A(X)$  than necessary, as follows

$$[D_M \varepsilon]_{W=0} \equiv \left[ \partial_M \varepsilon + \frac{1}{4} \omega_M^{ij} \Gamma_{ij} \varepsilon \right]_{W=0} = 0, \quad \left[ (\Gamma^M \varepsilon)_A (\partial_M \ln \Omega^{\frac{d-4}{d-2}}) \right]_{W=0} = 0. \quad (2.8)$$

In this case Eq.(1.7) is solved for  $U_A^Q = 0$  which, as mentioned following (1.7), could be chosen arbitrarily. Note that the second equation in (2.8) becomes trivial in the case of  $d+2 = 6$ ,

so it constrains  $\varepsilon_A$  only when  $d+2 = 12, 8, 5$  but not when  $d+2 = 6$ . The first equation requires a covariantly constant spinor  $[D_M \varepsilon]_{W=0} = 0$  in any of the curved backgrounds that obey Eq.(1.5). There are non-trivial backgrounds with covariantly constant spinors<sup>6</sup> so it is of interest to study those backgrounds that would be physically relevant in the applications of  $\text{SYM}_{10+2}^1$ .

For a more explicit solution in  $d+2$  dimensions we specialize further to the flat background described in footnote (1) which implies a constant spinor  $\partial_M \varepsilon_A = 0$  since  $\omega_M^{ij} = 0$ . We further take a special form for the dilaton  $\Omega = (c \cdot X)^{1-d/2}$  with a constant vector  $c_M$ , to satisfy Eq.(1.5). Then Eq.(2.8) becomes  $(d-4) c_M (\Gamma^M \varepsilon) = 0$ . By multiplying with another factor of  $c_M \Gamma^M$  we obtain the equation  $(d-4) c^2 \varepsilon_A = 0$ . Evidently for  $d+2 = 4$  the last equation puts no constraint on  $\varepsilon$  since it is trivially satisfied for any 4-component constant complex spinor  $\varepsilon_A$  (4 complex or 8 real fermionic parameters, so 8 supersymmetries which are part of  $\text{SU}(2, 2|1)$ , with  $\text{SU}(2, 2) = \text{SO}(4, 2)$ ). However, in  $d+2 = 12, 8, 5$  dimensions it requires a lightlike vector  $c^2 = 0$  with  $c_M (\Gamma^M \varepsilon) = 0$ . This has solutions only when half of the components of  $\varepsilon_A$  vanish. Thus, for example, in  $d+2 = 12$  dimensions, 16 out of the 32 real components of the constant SUSY parameter must vanish. Hence  $\text{SYM}_{10+2}^1$ , when taken with a constant SUSY spinor in a *flat background* in  $10+2$  dimensions, has at most 16 independent real parameters in  $\varepsilon_A$ , and hence 16 non-trivial supersymmetries.

For the more general backgrounds that obey (1.5) as well as Eq.(2.8) with covariantly constant spinors  $D_M \varepsilon = 0$ , a similar argument requires that  $(\partial_M \ln \Omega^{\frac{d-4}{d-2}})$  should be a lightlike vector (when  $d+2 \neq 6$ ) and therefore  $\varepsilon_A(X)$  still has at most 16 independent non-zero components for  $d+2 = 12$ . However, since these are  $X$ -dependent, the number of *constant* parameters in the 16 non-zero components of the spinor  $\varepsilon_A(X)$  may exceed 16 in some backgrounds.

These results hold for the special class of solutions of Eq.(1.7) that follow from the stronger requirements in Eq.(2.8). A similar result holds also for the general solutions as discussed in Appendix (B). However, when the background is curved, there are also cases with 32 supersymmetries, as in the example of compactification from  $10+2$  to  $4+2$  dimensions shown on the right side of Fig.1 and treated in section (III B).

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<sup>6</sup> For a discussion of covariantly constant spinors in non-trivial backgrounds see ref. [32], Eq.(15.1.3), and related discussion in chapter 15.

### III. SHADOWS AND COMPACTIFICATIONS

In this section we will show that  $\text{SYM}_{10+2}^1$  provides a higher dimensional source and new perspectives for the popular  $\text{SYM}_{9+1}^1$  and  $\text{SYM}_{3+1}^4$  that continue to be of intense interest in current research. We will use usual techniques of dimensional reduction as well as techniques of deriving shadows of 2T-physics [1][4][5][6][7] to obtain the lower dimensional theories.

#### A. Conformal shadow of $\text{SYM}_{10+2}^1$ gives $\text{SYM}_{9+1}^1$

We first briefly describe the result and then show how it is derived. We choose a set of coordinates  $X^M = (w, u, x^\mu)$  such that the function  $W(X)$  is simply  $W(X) = w$  in terms of the new coordinates. To see how such a basis can be chosen even in flat space see Appendix A. Here we also explain how the general background metric  $ds^2 = dX^M dX^N G_{MN}$  is brought to a basis that is convenient to generate the conformal shadow as in [6][7] while imposing  $w = 0$  as required by the delta function  $\delta(W(X))$  in the action. In the set of coordinates  $(w, u, x^\mu)$  we can solve all the kinematic constraints in Eqs.(1.5,2.4,2.3) for both the background and dynamical fields. We will show that by a series of gauge choices and solving the kinematic constraints we end up with the following shadow field configuration: The original fields  $A_M^a, F_{MN}^a, \lambda_A^a$  and  $\Omega, G_{MN}$  in  $d+2$  dimensions are then expressed in terms of the shadow fields at  $W(X) = w = 0$  as functions of the remaining coordinates  $u$  and  $x^\mu$  as follows

$$\begin{aligned} A_M^a(X) &= \begin{cases} A_\mu^a = A_\mu^a(x), \\ A_w = A_u = 0, \end{cases} & F_{MN}^a(X) &= \begin{cases} F_{\mu\nu}^a = F_{\mu\nu}^a(x), \\ F_{w\mu}^a = F_{u\mu}^a = F_{wu}^a = 0, \end{cases} \\ \lambda_A^a(X) &= \begin{pmatrix} \lambda_\alpha(x) \\ 0 \end{pmatrix} e^{(d-1)u}, & \Omega(X) &= e^{(d-2)u} \phi(x), \\ G_{MN}(X) &= \begin{cases} G_{\mu\nu} = e^{-4u} g_{\mu\nu}(x), \quad G_{wu} = -1, \\ G_{ww} = G_{w\mu} = G_{u\mu} = 0. \end{cases} \end{aligned} \quad (3.1)$$

The shadow fields  $A_\mu^a(x), \lambda_\alpha(x)$  form precisely the Yang-Mills supermultiplet in  $d = 10, 6, 4, 3$  dimensions in a *background* shadow spacetime described by  $g_{\mu\nu}(x), \phi(x)$ . Note that there are no Kaluza-Klein degrees of freedom since for example  $A_M^a(X) \rightarrow A_\mu^a(x)$ , and similarly for the other fields. Having solved all the kinematic constraints (which amounts to imposing  $\text{Sp}(2, R)$  invariance), our original action in  $d+2$  dimensions can now be reduced to the conformal shadow action in  $d$  dimensions that includes gravity coupled to a conformally

coupled dilaton  $\phi$  (with the wrong sign kinetic term) [6][7]

$$\begin{aligned} S &= S_{SYM} + \int d^d x \sqrt{-g} \left( \frac{d-2}{8(d-1)} \phi^2 R(g) + \frac{1}{2} \partial\phi \cdot \partial\phi \right), \\ S_{SYM} &= \int d^d x \sqrt{-g} \left( -\frac{1}{4g_{YM}^2} \phi^{2\frac{d-4}{d-2}} F_{\mu\nu}^a F_a^{\mu\nu} + i \bar{\lambda}^a \gamma^\mu D_\mu \lambda_a \right). \end{aligned} \quad (3.2)$$

The shadow dilaton can be fixed to a constant  $\phi(x) \rightarrow \phi_0$  by a Weyl transformation of all the fields<sup>7</sup>. Then the part  $S_{SYM}$  is recognized as the action in  $d = 10, 6, 4, 3$  dimensions for  $\text{SYM}_d^1$  in a curved background  $g_{\mu\nu}(x)$  and a constant dilaton  $\phi(x) = \phi_0$  with a dimensionful Yang-Mills coupling constant (dimensionless only for  $d = 4$ )

$$\hat{g}_{YM} = g_{YM} \phi_0^{-\frac{d-4}{d-2}}. \quad (3.3)$$

Here the covariant derivative  $D_\mu \lambda_a$  includes the Yang-Mills gauge field as well the spin connection  $\omega_\mu^{ab}$ , and  $\gamma^\mu \equiv e_a^\mu \gamma^a$  includes the vielbein  $e_a^\mu(x)$  associated with the general metric  $g_{\mu\nu}(x)$ . Of course, the well known flat case in which  $g_{\mu\nu}$  is fixed to the Minkowski metric  $\eta_{\mu\nu}$  and  $\phi$  is fixed to a constant, is a special case of the above.

The supersymmetry properties of the shadow action (3.2) in  $d = 3, 4, 6, 10$  dimensions, in the presence of gravity and the dilaton  $\phi(x)$  (but not yet supergravity), follow from the SUSY condition in  $d + 2$  dimensions (1.7), which is analyzed in detail in Appendix (B), including the conserved SUSY current. From that analysis we learn that this action is supersymmetric, without transforming  $g_{\mu\nu}(x), \phi(x)$ , but transforming only  $A_\mu^a$  and  $\lambda^a$  under SUSY, as follows

$$\delta_\varepsilon \lambda^a = \frac{i}{g_{YM}} \phi^{\frac{d-4}{d-2}} F_{\mu\nu}^a \gamma^{\mu\nu} \varepsilon, \quad \delta_\varepsilon A_\mu^a = -2 \phi^{-\frac{d-4}{d-2}} \bar{\varepsilon} \gamma_\mu \lambda^a + h.c., \quad (3.4)$$

provided the  $\text{SO}(d, 1)$  spinor SUSY parameter  $\varepsilon(x)$  satisfies the following conditions derived in Appendix B (treating  $g_{\mu\nu}, \phi$  as backgrounds, indices lowered/raised using  $g_{\mu\nu}$ )

$$D_\mu \varepsilon = \frac{1}{d} \gamma_\mu (\bar{\gamma} \cdot D \varepsilon) \quad \text{and} \quad (d-4) \bar{\gamma}^\mu D_\mu \left( \phi^{\frac{d}{d-2}} \varepsilon \right) = 0, \quad (3.5)$$

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<sup>7</sup> The local scaling, known as the Weyl symmetry, is a natural outcome of 2T-gravity [7]. It arises as a remnant of the general coordinate symmetry in the extra dimensions (there is no Weyl symmetry in the action in  $d + 2$  dimensions). Using this remnant local symmetry, the negative norm dilaton  $\phi(x)$  can be removed as a degree of freedom, thus insuring unitarity. Furthermore this Weyl gauge introduces Newton's gravitational constant in the conformal shadow. Note that, even though  $\phi(x)$  can be set to a constant by a Weyl gauge, the original dilaton field  $\Omega(X)$  still depends on the extra coordinate  $u$ , as given in Eq.(3.1). As discussed in [8] other Weyl gauge choices for the dilaton, which also remove the ghost  $\phi(x)$ , may be more convenient for certain useful applications of the shadows concept.

where  $D_\mu \varepsilon_\alpha(x) = \partial_\mu \varepsilon_\alpha(x) + \frac{1}{4} \omega_\mu^{ab}(x) (\gamma_{ab} \varepsilon(x))_\alpha$ . Note that the second equation is trivial for  $d = 4$ , so for  $d = 3, 6, 10$  there are two constraints on  $\varepsilon(x)$ , but for  $d = 4$  only one constraint. Here the spinors  $\varepsilon$  or  $\lambda$  have the following numbers of components (this is half of the  $\text{SO}(d, 2)$  spinor, i.e.  $\varepsilon_1$  as indicated in Eq.(B9))

$$\begin{aligned} d = 3 & : \text{ the spinor of } \text{SO}(2, 1) \text{ is real} = \text{a doublet of } \text{SL}(2, R), \\ d = 4 & : \text{ the Weyl spinor of } \text{SO}(3, 1) = \text{a complex doublet of } \text{SL}(2, C), \\ d = 6 & : \text{ the Weyl spinor of } \text{SO}(5, 1), \text{ a complex quartet.} \\ d = 10 & : \text{ the Weyl-Majorana spinor of } \text{SO}(9, 1) \text{ with 16 real components.} \end{aligned} \tag{3.6}$$

We emphasize that the spinor  $\varepsilon(x)$  is  $x$ -dependent, and thus may contain more than one set of constant spinor parameters. This number constant spinors, which determines the number of supersymmetries, will depend on the background  $g_{\mu\nu}(x), \phi(x)$  which in turn lead to the allowed solutions for  $\varepsilon(x)$  in Eq.(3.5).

For example, consider the  $d = 4$  flat space background  $g_{\mu\nu} = \eta_{\mu\nu}$  with  $\phi = \phi_0 = \text{a constant}$ . The solution of Eq.(3.5) is

$$d = 4, \text{ flat: } \varepsilon(x) = \varepsilon^{(0)} + x \cdot \gamma \varepsilon^{(1)}, \text{ with } \varepsilon^{(0)}, \varepsilon^{(1)} \text{ constant } \text{SL}(2, C) \text{ doublets.} \tag{3.7}$$

In this case  $\varepsilon^{(0)}$  corresponds to the usual supersymmetry parameter while  $\varepsilon^{(1)}$  corresponds to the superconformal transformation parameter. The closure of these transformations gives the global  $\text{SU}(2, 2|1)$  symmetry of  $\mathcal{N}=1$  super Yang-Mills theory in flat  $d = 4$ , which has 8 supersymmetries, namely the 8 real fermionic parameters in the two complex  $\text{SL}(2, C)$  doublets.

Repeating the same analysis for  $d = 10, 6, 3$ , still in the flat background, the first equation has the same form as (3.7), but the second equation in (3.5) eliminates  $\varepsilon^{(1)}$ , so that the solution is

$$d = 10, 6, 3, \text{ flat: } \varepsilon(x) = \varepsilon^{(0)}. \tag{3.8}$$

Hence, for  $d = 10$  there are only 16, not 32 supersymmetries in a flat background. However, in a curved background, in  $d = 10$ , the number could decrease or increase. For example, it is well known that when 6 of the 10 dimensions of  $\text{SYM}_{9+1}^1$  are compactified on a torus, the resulting theory  $\text{SYM}_{3+1}^4$  is  $\mathcal{N}=4$  super Yang-Mills theory in flat  $d = 4$ , which has  $\text{SU}(2, 2|4)$  symmetry, with 32 supersymmetries (as shown on the left branch in Fig.1).

A similar analysis for various fixed non-flat backgrounds  $g_{\mu\nu}(x), \phi(x)$  determines the number and nature of supersymmetries. Whatever those are, they correspond to the shadow

of the supersymmetries of the original theory of Eq.(1.1) in  $d+2$  dimensions in the presence of the (non-supersymmetric) backgrounds  $G_{MN}, \Omega, W$ .

Every shadow of the same theory - with the same original background  $G_{MN}, \Omega, W$  taken in various gauges and parameterizations of the  $d+2$  coordinates - will have the same global supersymmetry as already determined by the SUSY condition in  $d+2$  dimensions (1.7). The shadows alluded to in this discussion are sketched in Fig.1. The same SUSY would take different non-linear (possibly hidden) forms in terms of the coordinates in various shadows. These shadows are all dual to each other as they retain the information of the original theory holographically. One unchanging aspect under the dualities is the global symmetry; in this case this includes the SUSY determined by (1.7).

### 1. Technical details

In this subsection we show how the results of section III A are derived for the conformal shadow. As discussed in [6][7], we choose a convenient set of coordinates  $X^M = (w, u, x^\mu)$ , such that  $W(X) = w$ , in terms of which we will express the solutions of the kinematic equations (1.5) that restrict the 2T geometry. See footnote (1) for another form of the geometry in Cartesian coordinates. It is assumed that this set of coordinates can be chosen by coordinate reparameterizations. For example, if the initial spacetime metric  $G_{MN}$  is the flat metric  $\eta_{MN}$  in  $d+2$  dimensions, the appropriate change of coordinates is given in Appendix A.

We start with the solution of the kinematics for the background geometry (1.5) as given in [7]. The results include the following properties of  $V_M \equiv \frac{1}{2}\partial_M W$ , at any  $w$ ,

$$\begin{aligned} W = V^M V_M = w, \quad V_M = \left(\frac{1}{2}, 0, 0\right)_M, \quad V^M = \left(2w, -\frac{1}{2}, 0\right)^M, \\ V_i = E_i^M V_M = \left(\frac{1}{2}, -w, 0\right)_i, \quad V = V_i \Gamma^i = \left(\frac{1}{2}\Gamma^{-'} - w\Gamma^{+'}\right) = \begin{pmatrix} 0 & -i\sqrt{2}w \\ \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \end{aligned} \quad (3.9)$$

The metric  $G_{MN}(X)$  and vielbein  $E_M^i(X)$  that satisfy (1.5) are given in terms of a general

$g_{\mu\nu}(x, we^{4u})$  or  $e_a^\mu(x, we^{4u})$ , at any  $w$ , as follows

$$G_{MN} = \begin{array}{c} M \backslash N \\ w \quad u \quad \nu \\ \left( \begin{array}{ccc} 0 & -1 & 0 \\ -1 & -4w & 0 \\ 0 & 0 & e^{-4u} g_{\mu\nu} \end{array} \right) \end{array}, \quad G^{MN} = \begin{array}{c} M \backslash N \\ w \quad u \quad \nu \\ \left( \begin{array}{ccc} 4w & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & e^{4u} g^{\mu\nu} \end{array} \right) \end{array}, \quad (3.10)$$

and

$$E_M^i = \begin{array}{c} M \backslash i \\ w \quad u \quad \mu \\ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 2w & 1 & 0 \\ 0 & 0 & e^{-2u} e_\mu^a \end{array} \right) \end{array}, \quad E_i^M = \begin{array}{c} i \backslash M \\ -' \quad +' \quad a \\ \left( \begin{array}{ccc} 1 & 0 & 0 \\ -2w & 1 & 0 \\ 0 & 0 & e^{2u} e_a^\mu \end{array} \right) \end{array}, \quad (3.11)$$

while the volume element is

$$d^{d+2} X \sqrt{G} \delta(W) = (d^d x \, du \, dw) e^{-2du} \sqrt{-g} \delta(w). \quad (3.12)$$

The affine connection  $\Gamma_{MN}^P$ , spin connection  $\omega_M^{ij}$  and curvature  $R_{MNP}^Q$  are computed in [7]. In this paper we will only need the expressions for  $\Gamma_{wN}^P, \Gamma_{uN}^P$  and  $\omega_w^{ij}, \omega_u^{ij}, \omega_\mu^{ij}$  taken from [7] as follows

$$\Gamma_{wN}^P = \begin{array}{c} N \backslash P \\ w \quad u \quad \lambda \\ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} g^{\lambda\sigma} \partial_w g_{\sigma\nu} \end{array} \right) \end{array}, \quad \Gamma_{uN}^P = \begin{array}{c} N \backslash P \\ w \quad u \quad \lambda \\ \left( \begin{array}{ccc} 2 & 0 & 0 \\ 8w & -2 & 0 \\ 0 & 0 & -2\delta_\nu^\lambda + 2w g^{\lambda\sigma} \partial_w g_{\sigma\nu} \end{array} \right) \end{array}. \quad (3.13)$$

$$\omega_w^{ij} = \begin{array}{c} i \backslash j \\ -' \quad +' \quad b \\ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} e^{\mu[a} \partial_w e_\mu^{b]} \end{array} \right) \end{array}, \quad \omega_u^{ab} = \begin{array}{c} i \backslash j \\ -' \quad +' \quad b \\ \left( \begin{array}{ccc} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2w e^{\mu[a} \partial_w e_\mu^{b]} \end{array} \right) \end{array}. \quad (3.14)$$

and

$$\omega_\mu^{ij} = \begin{array}{c} i \backslash j \\ -' \quad +' \quad b \\ \left( \begin{array}{ccc} 0 & 0 & e^{-2u} (-2e_\mu^b + w e^{b\sigma} \partial_w g_{\mu\sigma}) \\ 0 & 0 & \frac{e^{-2u}}{2} e^{b\nu} \partial_w g_{\mu\nu} \\ e^{-2u} (2e_\mu^a - w e^{a\sigma} \partial_w g_{\lambda\sigma}) & -\frac{e^{-2u}}{2} e^{a\sigma} \partial_w g_{\mu\sigma} & \omega_\mu^{ab}(e) \end{array} \right) \end{array} \quad (3.15)$$



where  $\omega_\mu^{ab}(e)$  is the standard spin connection constructed from the vielbein  $e_\mu^a$  in  $d$  dimensions. It is interesting that all dependence on  $x^\mu$  and  $we^{4u}$  drops out in the following combination of connections

$$V^M \Gamma_{MN}^P = 2w \Gamma_{wN}^P - \frac{1}{2} \Gamma_{uN}^P = \delta_M^P - 2\delta_w^P \delta_M^w, \quad V^M \omega_M^{ij} = 2w \omega_w^{ij} - \frac{1}{2} \omega_u^{ij} = -\delta_+^{[i} \delta_-^{j]}. \quad (3.16)$$

In this basis the kinematic equations for the dilaton and the SUSY parameter Eqs.(1.5,2.3) simplify to  $(2w\partial_w - \frac{1}{2}\partial_u + \frac{d-2}{2})\Omega(X) = 0$  and  $(2wD_w - \frac{1}{2}D_u)\varepsilon(X) = (2w\partial_w - \frac{1}{2}\partial_u + \frac{1}{2}\Gamma^{+-'})\varepsilon(X) = 0$  respectively. These restrict the  $u, w$  dependence of the dilaton and the SUSY parameter as follows

$$\Omega(X) = e^{(d-2)u} \hat{\Omega}(x, we^{4u}), \quad \varepsilon(X) = \exp\left(u\Gamma^{+-'}\right) \hat{\varepsilon}(x, we^{4u}). \quad (3.17)$$

The backgrounds  $G_{MN}, E_M^i, \omega_M^{ij}, \Gamma_{MN}^P, \Omega$  occur in our action without further derivatives with respect to  $u$  or  $w$ , while there is a delta function  $\delta(w)$  in the volume element (3.12). So, in (3.10-3.17), considering a Taylor expansion in powers of  $we^{4u}$ , we must keep only the zeroth order terms since all higher order terms in  $w$  drop out due to  $w^p \delta(w) = 0$  for integers  $p \geq 1$ .

Now we turn to the solution of the kinematic constraints (2.4) for the dynamical fields  $A_M^a, \lambda_A^a$ . We will follow a procedure similar to sections 4B and 4C in [1] except for generalizing to curved space and higher dimensions. We will work in the Yang-Mills gauge given by  $V \cdot A^a = 2wA_w^a - \frac{1}{2}A_u^a = 0$ . In this gauge there remains a subset of Yang-Mills gauge symmetry which does not change the gauge  $V \cdot A^a = 0$ . For this subset the gauge parameters  $\Lambda^a(X)$  satisfy  $0 = V \cdot \delta_\Lambda A^a = V \cdot D\Lambda^a = V \cdot \partial\Lambda^a = (2w\partial_w - \frac{1}{2}\partial_u)\Lambda^a$ . So the remaining Yang-Mills gauge symmetry has the form

$$\Lambda^a(X) = \hat{\Lambda}^a(x, we^{4u}), \quad (3.18)$$

with  $\hat{\Lambda}^a$  an arbitrary function of  $x$  and  $we^{4u}$ . This will be used for further Yang-Mills gauge fixing.

In the gauge  $V \cdot A^a = 0$  the interaction terms with the Yang-Mills field disappear in the kinematic constraints (2.4)  $0 = V^M F_{MN}^a = (V \cdot \nabla + 1)A_N^a$ , where we have also used  $\nabla_M V_N = G_{MN}$  (see 1.5) to pass  $V$  through  $\partial$ . So, we have

$$0 = (V \cdot \nabla + 1)A_N^a = \left(2w\partial_w - \frac{1}{2}\partial_u + 1\right)A_N^a - \left(2w\Gamma_{wN}^P - \frac{1}{2}\Gamma_{uN}^P\right)A_P^a. \quad (3.19)$$

Taking into account Eq.(3.16), the solution of this kinematic equation for  $A_M^a(X)$  is

$$A_w^a(X) = e^{4u} \hat{A}_w^a(x, we^{4u}), \quad A_u^a(X) = \hat{A}_u^a(x, we^{4u}), \quad A_\mu^a(X) = \hat{A}_\mu^a(x, we^{4u}). \quad (3.20)$$

Similarly, the kinematic constraint for the spinor is solved as follows (taking into account Eq.(3.16))

$$0 = \left( V \cdot D + \frac{d}{2} \right) \lambda^a = \left( 2wD_w - \frac{1}{2}D_u + \frac{d}{2} \right) \lambda^a \quad (3.21)$$

$$\Rightarrow \lambda^a = \exp \left( ud + u\Gamma^{+'-'} \right) \hat{\lambda}^a(x, we^{4u}). \quad (3.22)$$

Next we recall that the action (1.1) and the SUSY current (2.2) are gauge invariant under the 2T gauge transformations<sup>8</sup>  $\delta_s A_M^a = W s_M^a(X)$  and  $\delta_\kappa \lambda^a = V \kappa_1^a(X) + W \kappa_2^a(X)$ , with local bosonic parameters  $s_M^a(X)$  and fermionic parameters  $\kappa_{1A}^a(X), \kappa_{2A}^a(X)$ . From this we deduce that the kinematically constrained fields above transform under these gauge symmetries as follows

$$\begin{aligned} \delta_s \hat{A}_M^a &= we^{4u} \hat{s}_M^a(x, we^{4u}), \\ \delta_\kappa \hat{\lambda}^a &= \left( \frac{1}{2} \Gamma^{-'} - we^{4u} \Gamma^{+'} \right) \hat{\kappa}_1^a(x, we^{4u}) + we^{4u} \hat{\kappa}_2^a(x, we^{4u}). \end{aligned} \quad (3.23)$$

This is enough gauge symmetry at any  $w$  to gauge fix  $\hat{A}_M^a(x, we^{4u}), \hat{\lambda}_A^a(x, we^{4u})$  to functions of only  $x$  (it may be helpful for the reader to contemplate an expansion in powers of  $we^{4u}$ ), while eliminating half of the spinor degrees of freedom with the gauge choice<sup>9</sup>  $\Gamma^{+'} \hat{\lambda}^a = 0$

$$\hat{A}_M^a(x, we^{4u}) \rightarrow A_M(x), \quad \hat{\lambda}_A^a(x, we^{4u}) \rightarrow \begin{pmatrix} \lambda_A^a(x) \\ 0 \end{pmatrix}, \quad \overline{\hat{\lambda}}^a(x, we^{4u}) \rightarrow i \left( 0 \quad \overline{\lambda}^a(x) \right). \quad (3.24)$$

Now, recall that  $V \cdot A = 0$  implied that  $A_u(X) = 2wA_w(X)$ . Inserting the forms in (3.20) this gives  $\hat{A}_u = 2we^{4u} \hat{A}_w(x, we^{4u})$ . We can now use the remaining Yang-Mills symmetry in (3.18) to fix further the gauge  $\hat{A}_w(x, we^{4u}) = 0$ , which then also makes  $\hat{A}_u = 0$ .

<sup>8</sup> The gauge symmetry of the action with fields completely off shell involves more complicated transformation rules than the one shown here, and in that case the parameters  $s_M^a(X), \kappa_{1A}^a(X), \kappa_{2A}^a(X)$  are arbitrary. However, in the present case the fields  $A_M^a(X), \lambda_A^a(X)$  in (3.20,3.22) already satisfy the kinematic constraints (2.4), so the corresponding local parameters  $s_M^a(X), \kappa_{1A}^a(X), \kappa_{2A}^a(X)$  must be specialized to a subset that is consistent with the kinematic constraints. Hence we must have  $s_w^a(X) = \hat{s}_w^a(x, we^{4u})$ ,  $s_u^a(X) = e^{4u} \hat{s}_u^a(x, we^{4u})$  and  $s_\mu^a(X) = e^{4u} \hat{s}_\mu^a(x, we^{4u})$  and similarly  $\kappa_1^a(X) = \exp \left( u\Gamma^{+'-'} + u(d+2) \right) \hat{\kappa}_1^a(x, we^{4u})$ , and  $\kappa_2^a(X) = \exp \left( u\Gamma^{+'-'} + u(d+4) \right) \hat{\kappa}_2^a(x, we^{4u})$ . Under such specialized transformations, as in the more general case, the kinematic constraints (2.4) on  $A_M^a(X), \lambda_A^a(X)$  as well as the action or the dynamical equations (2.5) are invariant.

<sup>9</sup> Here we use explicit gamma matrices that satisfy  $\Gamma^i \bar{\Gamma}^j + \Gamma^j \bar{\Gamma}^i = 2\eta^{ij} : \Gamma^{+'} = \begin{pmatrix} 0 & -i\sqrt{2} \\ 0 & 0 \end{pmatrix}, \Gamma^{-'} = \begin{pmatrix} 0 & 0 \\ -i\sqrt{2} & 0 \end{pmatrix}, \Gamma^\mu = \begin{pmatrix} \bar{\gamma}^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix}$  with  $\gamma^\mu = (-1, \gamma^i), \bar{\gamma}^\mu = (1, \gamma^i)$  and  $\bar{\Gamma}^{+'} = \begin{pmatrix} 0 & -i\sqrt{2} \\ 0 & 0 \end{pmatrix}, \bar{\Gamma}^{-'} = \begin{pmatrix} 0 & 0 \\ -i\sqrt{2} & 0 \end{pmatrix}, \bar{\Gamma}^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\bar{\gamma}^\mu \end{pmatrix}$ . In this basis the conjugate of  $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$  is given by  $\bar{\Psi} = i(\bar{\psi}_2, \bar{\psi}_1)$ . See appendix A in [2].

All of the above steps solve the kinematic constraints at any  $w$ . Having taken into account all derivatives with respect to  $w$ , now we can safely set  $w = 0$  on account that the volume element in the action contains the delta function  $\delta(w)$  as in (3.12). So, the above fields now are taken at  $w = 0$ , yielding only functions of spacetime  $x^\mu$  in  $d$  dimensions. The  $u$  dependence of all fields is explicit as in (3.1), and after inserting them into the action one finds that all  $u$  dependence of the Lagrangian  $L_{SYM}$  cancels out against the  $u$  dependence of the volume element (3.12), leaving an action density that is independent of  $u$ . Then the integral over  $u$  in the action is an overall infinite factor which is absorbed into the overall renormalization constant  $K$  in front of the action (1.1). Equivalently, this is a renormalization of the Planck constant in the path integral formalism.

In summary, by a series of gauge choices and solving kinematic constraints we end up with the configurations in Eq.(3.1) which describe the shadow fields in  $d$  dimensions. Inserting these in the original action (1.1) we obtain the results summarized in the shadow action (3.2) and the comments that follow it.

## B. Dimensional reduction $\text{SYM}_{10+2}^1 \rightarrow \text{SYM}_{4+2}^4 \rightarrow \text{shadow SYM}_{3+1}^4$

Let us now consider the reduction  $10+2 \rightarrow 4+2$  by taking the fields as a functions of the coordinates

$$X^M = (x^m, y^I), \quad \begin{cases} x^m \text{ vector of } \text{SO}(4, 2), \\ y^I \text{ vector of } \text{SO}(6). \end{cases} \quad (3.25)$$

We are aiming for a metric  $G_{MN}(x, y)$  consistent with  $\text{SO}(4, 2) \times \text{SO}(6)$  symmetry, but the overall metric need not be flat in  $10+2$  dimensions. In fact we will see that to recover  $\text{SYM}_{4+2}^1$  via compactification, with 32 supersymmetries, the metric  $G_{MN}(x, y)$  cannot be flat.

### 1. Background consistent with homothety and SUSY

We take a metric and vielbein of the form

$$G_{MN} = \begin{matrix} & n & J \\ m & \eta_{mn} & 0 \\ I & 0 & a^2(x, y) \delta_{IJ} \end{matrix}, \quad E_M^i = \begin{matrix} & \alpha & a \\ m & \delta_m^\alpha & 0 \\ I & 0 & a(x, y) \delta_I^a \end{matrix}. \quad (3.26)$$

which is flat in  $4 + 2$  dimensions  $x^m$ , and conformally flat in the extra 6 dimensions  $y^I$  due to the warp factor  $a^2(x, y)$ . Furthermore, by choosing coordinates such that  $W(X) = x^2$  we get

$$V^M(x, y) = \frac{1}{2} G^{MN} \partial_M W = (x^m, 0) \text{ and } V^i = V^M E_M^i = (\delta_m^\alpha x^m, 0). \quad (3.27)$$

Next impose the homothety conditions for the metric and vielbein (1.5)

$$\mathcal{L}_V G_{MN} = 2G_{MN}, \quad \mathcal{L}_V E_M^i = E_M^i. \quad (3.28)$$

Specializing  $M \rightarrow m, I$  we learn that the conformal factor  $a(x, y)$  must be homogeneous and satisfies the equation

$$x \cdot \partial \ln a = 1, \text{ or } a(tx, y) = ta(x, y). \quad (3.29)$$

Similarly, the homothety condition (1.5) for the dilaton  $\Omega$  reduces to  $(x \cdot \partial + \frac{d-2}{2}) \Omega = 0$ , which requires a homogeneous dilaton

$$\Omega(tx, y) = t^{-4} \Omega(x, y) \text{ for } d + 2 = 12. \quad (3.30)$$

The spin connection  $\omega_M^{ij}$  has to reproduce the  $E_M^i$  above through  $E_M^i = D_M V^i$ . Hence the spin connection can be taken as

$M \setminus ij$	$\alpha\beta$	$\beta a$	$ab$
$\omega_m^{ij} =$	0	0	0
$\omega_I^{ij} =$	0	$\omega_I^{\beta a} = \delta_I^a \delta^{\beta m} \partial_m \ln a$	$\omega_I^{ab} = \delta_I^{[a} \delta^{b]J} (\partial_J \ln a)$

(3.31)

With these  $\omega_M^{ij}(x, y)$  the torsion tensor vanishes, as it should,  $T_{MN}^i = D_{[M} E_{N]}^i = \partial_{[M} E_{N]}^i + \omega_{[M}^{ij} E_{N]j} = R_{MN}^{ij} V_j = 0$ .

## 2. Reduction of the 10+2 action to 4+2

Now consider the action. We are aiming to obtain the dimensionally reduced action to coincide with  $\text{SYM}_{4+2}^4$  whose action was given in [3]. The Yang-Mills field  $A_M(X) = (A_m, A_I)(x)$  is taken independent of  $y^I$  due to the dimensional reduction. Hence in constructing  $F_{MN} = (F_{mn}, F_{mI}, F_{IJ})$  all derivatives with respect to  $y^I$  are dropped, so that

$$F_{mI} = D_m A_I = \partial_m A_I + A_m \times A_I; \quad F_{IJ} = A_I \times A_J, \quad (3.32)$$

where  $A_m \times A_I$  is a short hand notation for the adjoint action of the Yang-Mills group,

$$(A_m \times A_I)_a \equiv f_{abc} A_m^b A_I^c, \text{ etc.} \quad (3.33)$$

Then, the Yang-Mills term in Eq.(1.2) becomes

$$\begin{aligned} & -\frac{1}{4g_{YM}^2} \delta(W) \Omega^{3/2} \sqrt{G} G^{MP} G^{NQ} F_{MN} F_{NQ} \\ & = -\frac{1}{4g_{YM}^2} \delta(x^2) \Omega^{3/2} a^6 \left( (F_{mn})^2 + 2a^{-2} (D_m A_I)^2 + a^{-4} (A_I \times A_J)^2 \right) \\ & = -\frac{1}{4g_{YM}^2} \delta(x^2) \Omega^{3/2} a^6 \left( (F_{mn})^2 + 2 \left( D_m \frac{A_I}{a} + \frac{A_I}{a} \partial_m \ln a \right)^2 + \left( \frac{A_I}{a} \times \frac{A_J}{a} \right)^2 \right) \end{aligned} \quad (3.34)$$

Here we will identify  $A_I/a$  with the six scalar fields in  $\text{SYM}_{4+2}^4$

$$\phi_I(x) = \frac{1}{g_{YM}} \frac{A_I(x)}{a(x, y)}, \text{ vector of SO}(6). \quad (3.35)$$

Since  $\phi_I(x)$  must be independent of  $y^I$  we must take the warp factor  $a(x, y)$  independent of  $y^I$ . The kinetic term for  $\phi_I(x)$  coming from the reduction from 10+2 contains the form

$$D_m \left( \frac{A_I}{ag_{YM}} \right) + \left( \frac{A_I}{ag_{YM}} \right) \partial_m \ln a = (D_m \phi_I + \phi_I \partial_m \ln a). \quad (3.36)$$

So the kinetic term for  $\phi_I$  in Eq.(3.34) becomes (for all contractions over  $m$  we use the flat  $\text{SO}(4, 2)$  metric)

$$\begin{aligned} & -\frac{1}{2} \delta(x^2) (D_m \phi_I + \phi_I \partial_m \ln a)^2 \\ & = -\frac{1}{2} \delta(x^2) \left( (D_m \phi_I)^2 + \partial_m \ln a \partial_m \phi^2 + \phi^2 (\partial_m \ln a)^2 \right) \\ & = \left\{ \begin{aligned} & +\frac{1}{2} \delta(x^2) \phi_I D_m^2 \phi_I + \frac{1}{2} \phi^2 \left\{ \begin{aligned} & \delta'(x^2) (-2 + 2x \cdot \partial \ln a) \\ & + \delta(x^2) (\partial_m^2 \ln a - (\partial_m \ln a)^2) \end{aligned} \right\} \\ & + \frac{1}{2} \partial^m [-\delta(x^2) \phi_I D_m \phi_I + \delta'(x^2) x_m \phi^2 - \delta(x^2) (\partial_m \ln a) \phi^2] \end{aligned} \right\} \end{aligned} \quad (3.37)$$

Here the last term is a total derivative and can be dropped in the action. To obtain this form we used  $\partial_m x^m = 6$  and  $x \cdot \partial \delta'(x^2) = -4\delta'(x^2)$ . Using  $x \cdot \partial \ln a = 1$  in Eq.(3.29) the coefficient of  $\delta'(x^2)$  vanishes,  $(-2 + 2x \cdot \partial \ln a) = 0$ . Hence the kinetic term for the scalar field is

$$\delta(x^2) \Omega^{3/2} a^6 \left\{ \frac{1}{2} \phi_I D^2 \phi_I + \frac{1}{2} \phi_I^2 (\partial_m^2 \ln a - (\partial_m \ln a)^2) \right\}. \quad (3.38)$$

The last term could be interpreted as a coupling to a background curvature in 4+2 dimensions, but we will continue here under the assumption that the 4+2 background is

flat since we are trying to compare to the  $\text{SYM}_{4+2}^4$  in [3]. Hence we need to impose  $\partial_m^2 \ln a - (\partial_m \ln a)^2 = 0$ . So the coefficient of  $\phi^2 \delta(x^2)$  vanishes only when  $a(x)$  satisfies the following solution

$$\partial^2 \ln a - (\partial_m \ln a)^2 = 0 \rightarrow a(x) = x \cdot b \text{ and } b^m b_m = 0. \quad (3.39)$$

To get the normalizations of the first and second terms in Eq.(3.34) to coincide with [3] we must also have  $\Omega^{3/2} a^6 = 1$ . Hence  $a(x, y), \Omega(x, y)$  should both be independent of  $y^I$  and related to each other as

$$a(x) = \Omega^{-\frac{1}{4}}(x) = x \cdot b. \quad (3.40)$$

A tricky term in the  $\text{SYM}_{4+2}^4$  action is the kinetic term for the scalar that has the form

$$\frac{1}{2} \delta(x^2) \phi_I D_m^2 \phi_I \quad (3.41)$$

rather than  $-\frac{1}{2} \delta(x^2) \eta^{mn} D_m \phi_I D_n \phi_I$ . These are not the same because an integration by parts involves a difference term proportional to  $\delta'(x^2)$ . This form of the kinetic term for scalars is required by both the 2T gauge symmetries and the SUSY symmetry in  $4+2$  dimensions (for the most general form permitted in the presence of curved backgrounds see [10]). Then, with the form of  $a(x)$  in Eq.(3.40), we obtain the correct kinetic term for the scalars

$$-\frac{1}{2} \delta(x^2) (D_m \phi_I + \phi_I \partial_m \ln a)^2 = \frac{1}{2} \delta(x^2) \phi_I D^2 \phi_I + \text{total derivative}. \quad (3.42)$$

Note that the constant vector  $b_m$  has disappeared from all terms. So there is no preferred direction in the resulting action and therefore there is an  $\text{SO}(4, 2) \times \text{SO}(6)$  symmetry.

Next we consider the fermions. For correct normalization, the fermion must be taken as

$$\lambda_A(x, y) = a^{-3}(x) \psi_A(x), \quad A = 1, 2, \dots, 32. \quad (3.43)$$

Then the fermion action becomes

$$\frac{i}{2} \delta(W) \sqrt{G} \bar{\lambda} V D \lambda + h.c \quad (3.44)$$

$$= \frac{i}{2} \delta(x^2) \sqrt{G} a^{-3} \bar{\psi} x \left( \Gamma^m D_m + \Gamma^a E_a^I (\partial_I + \omega_I + A_I \times) \right) (\psi a^{-3}) + h.c \quad (3.45)$$

$$= \frac{i}{2} \delta(x^2) (a^6 a^{-6}) \bar{\psi} x \left( \begin{array}{c} \Gamma^m D_m + \Gamma^m \partial_m \ln a^{-3} \\ + \frac{1}{a} \Gamma^I \left( \begin{array}{c} \partial_I \ln a^{-3} - \frac{1}{2} \Gamma_\beta \Gamma_a \omega_I^{\beta a} \\ + \frac{1}{4} \Gamma_{cd} \omega_I^{cd} + A_I \times \end{array} \right) \end{array} \right) \psi(x) + h.c \quad (3.46)$$

After taking into account that  $a$  is independent of  $y^I$  we can drop  $\omega_I^{cd} = \delta_I^{[c} \delta^{d]J} (\partial_J \ln a) = 0$  and write  $\omega_I^{\beta a} = \delta_I^a \delta^{\beta m} \partial_m \ln a$ , we note that

$$-3\Gamma^m \partial_m \ln a + \frac{1}{2} \Gamma^I \Gamma^I \Gamma^m \partial_m \ln a = 0. \quad (3.47)$$

Hence we get the correct kinetic term for fermions that agrees with the expected form for  $\text{SYM}_{4+2}^4$  in agreement with [3]

$$\frac{i}{2} \delta(W) \sqrt{G} \bar{\lambda} V D \lambda = \frac{i}{2} \delta(x^2) \bar{\psi} [x (\Gamma^m D_m + g \Gamma^I \phi_I \times)] \psi(x)$$

Putting together the result of the reduction, and dropping the total derivative in Eq.(3.37), we obtain the reduced Lagrangian

$$L_{SYM}(x, y) = \delta(x^2) \left( -\frac{1}{4g_{YM}^2} (F_{mn})^2 + \frac{1}{2} \phi_I D^2 \phi_I - \frac{g_{YM}^2}{4} (\phi_I \times \phi_J)^2 \right. \\ \left. + \frac{i}{2} \bar{\psi} [x (\bar{\Gamma}^m D_m + g_{YM} \bar{\Gamma}^I \phi_I \times)] \psi(x) + h.c \right) \quad (3.48)$$

The  $y$  integration over a compact space is an overall trivial factor that can be absorbed into the normalization  $K$  in the original action (1.1). We know from [3] that  $\text{SYM}_{4+2}^4$  is the 2T-physics parent of  $\text{SYM}_{3+1}^4$ , hence we have established the connections shown with arrows on the right hand side of Fig.1.

In this last form the fermions are still retaining the 10+2 notation for the 32  $\lambda$ 's as the spinor of  $\text{SO}(10, 2)$ , while the gamma matrices  $\Gamma^m, \Gamma^I$  are also  $32 \times 32$  matrices, thus *showing their 10+2 dimensional origin*. To relate to the spinors in  $4+2$  dimensions and to display the  $\mathcal{N} = 4$  supersymmetry we must express the 32-spinor in an  $\text{SU}(2, 2) \times \text{SU}(4)$  basis as in [3]. This is a technical point in group theory but may be useful to show it explicitly as in Appendix (C).

Using the notation of  $32 \times 32$  gamma matrices provided in Appendix (C), it is straightforward to show that the fermion kinetic term in the  $\text{SO}(10, 2)$  notation with the 32  $\lambda$ 's is rewritten correctly in the  $\text{SO}(4, 2) \times \text{SO}(6) = \text{SU}(2, 2) \times \text{SU}(4)$  basis in agreement with Eq.(4.1) in [3] (the Yang-Mills group adjoint label  $a$  is now shown explicitly below, while the label  $r$  is for the 4 of  $\text{SU}(4)$  as defined in the appendix)

$$L^{\mathcal{N}=4}(x) = \delta(x^2) \left\{ -\frac{1}{4g_{YM}^2} F_{mn}^a F_a^{mn} + \frac{1}{2} \phi_I^a D^m D_m \phi_I^a - \frac{g_{YM}^2}{4} \sum (f_{abc} \phi_I^b \phi_J^c)^2 \right. \\ \left. + \frac{i}{2} [\bar{\psi}^{ar} x \bar{D} \psi_r^a + g_{YM} f_{abc} (\psi_r^a C \bar{x} \psi_s^b) (\bar{\gamma}^I)^{rs} \phi_I^c] + h.c. \right\} \quad (3.49)$$

In [3] it is shown how to rewrite the kinetic and potential energies of the six real scalars  $\phi_I^a$  in an  $\text{SU}(4)$  antisymmetric pseudo-complex matrix notation  $\varphi_{rs} = (\gamma^I)_{rs} \phi_I^c = \frac{1}{2} \varepsilon_{rstu} \bar{\varphi}^{tu}$ .

This  $SU(4)$  notation displays the *linearly realized*  $SU(2, 2|4)$  *supersymmetry* of the  $SYM_{4+2}^4$  theory directly in  $4 + 2$  dimensions. This is the origin of the superconformal symmetry that is *non-linearly* realized in the conformal shadow in the form of the conventional  $SYM_{3+1}^4$  theory shown at the bottom of Fig.1.

#### IV. M(ATRix) THEORY AS DIMENSIONALLY REDUCED $SYM_{10+2}^1$

It is well known that M(atrrix) theory in  $9+1$  dimensions is constructed by compactifying  $SYM_{9+1}^1$  [26]-[31]. Since we have already shown in section (III A) that  $SYM_{9+1}^1$  is a shadow of  $SYM_{10+2}^1$ , it is already evident that  $SYM_{10+2}^1$  is the 2T-physics source for M(atrrix) theory in  $9+1$  dimensions. In this section we want to make this connection to M(atrrix) theory directly from  $SYM_{10+2}^1$  without having to first go through the shadow  $SYM_{9+1}^1$ . Since this is the first direct link between 2T-physics and M-theory we want to make the connection as clear as possible.

The starting point is the action  $S_{SYM}$  in Eq.(1.2). Consider at first the Yang-Mills part<sup>10</sup> for the case  $d + 2 = 12$

$$L_{YM} = -\frac{1}{4g_{YM}^2} \sqrt{G} \delta(W) \Omega^{\frac{3}{2}} \frac{1}{2} Tr(F_{MN} F_{NQ}) G^{MP} G^{NQ} \quad (4.1)$$

We split the  $10 + 2$  coordinates  $X^M$  into two parts  $x^\mu \sim (9 + 1)$  and  $\sigma^m \sim (1 + 1)$

$$X^M = (\sigma^m, x^\mu). \quad (4.2)$$

We take all the fields to be independent of  $x^\mu$ , so that they depend only on  $\sigma^m$ . Then the Yang-Mills field strength  $F_{MN}$  splits into three parts,  $F_{MN} = (F_{mn}, F_{m\nu}, F_{\mu\nu})$ . Since all derivatives with respect to  $x^\mu$  are dropped, we have (where  $\partial_m \equiv \partial/\partial\sigma^m$ )

$$F_{mn} = \partial_m A_n - \partial_n A_m - i[A_m, A_n], \quad (4.3)$$

$$F_{m\mu} = D_m A_\mu = \partial_m A_\mu - i[A_m, A_\mu], \quad F_{\mu\nu} = -i[A_\mu, A_\nu], \quad (4.4)$$

<sup>10</sup> We are now writing the Yang-Mills group in matrix version instead of using the adjoint index  $a$ . The relation between the two is  $A_M = A_M^a t_a$  where  $t_a$  is a hermitian matrix representation in the fundamental representation of the group  $G$ . Then  $[A_M, A_N] = it^a (f_{abc} A_M^b A_N^c)$ , and  $t_a$  is normalized as  $Tr(t_a t_b) = 2\delta_{ab}$ .



We also take  $W(\sigma), \Omega(\sigma)$  as well as the metric  $G_{MN}(\sigma)$  to be only a function of  $\sigma^m$  and of the form

$$G_{MN} = \begin{pmatrix} g_{mn}(\sigma) & 0 \\ 0 & a^2(\sigma) \eta_{\mu\nu} \end{pmatrix}, \quad \sqrt{G} = \sqrt{g} a^{10}. \quad (4.5)$$

Then, the homothety conditions on the geometry Eqs.(1.5) are satisfied with the following forms

$$V_M = (v_m(\sigma), 0)_M, \quad V^M = (v^m, 0)^M, \quad v_m = \frac{1}{2} \partial_m W, \quad v^m = g^{mn} v_n. \quad (4.6)$$

$$0 = (v \cdot \partial - 1) a(\sigma), \quad (v \cdot \partial + 4) \Omega(\sigma), \quad \mathcal{L}_v g_{mn} = 2g_{mn}, \quad (4.7)$$

where  $\mathcal{L}_v$  is the Lie derivative with respect to the two dimensional vector  $v^m(\sigma)$  defined above. We see from the last line that it is consistent to take the warp factor  $a(\sigma)$  as a function of  $\Omega(\sigma)$  just as in the previous section Eq(3.40), but now as a general function of  $\sigma$ ,

$$a(\sigma) = \Omega^{-1/4}(\sigma). \quad (4.8)$$

Inserting these forms in the Yang-Mills action, and using  $\sqrt{G} \Omega^{3/2} = \sqrt{g} \Omega^{3/2} a^{10} = \sqrt{g} a^4$ , we obtain the following reduced form

$$L_{YM} = -\frac{1}{4g_{YM}^2} \delta(W) \sqrt{G} \Omega^{\frac{3}{2}} \frac{1}{2} Tr(F_{MN} F_{PQ}) G^{MP} G^{NQ} \quad (4.9)$$

$$= -\frac{1}{4g_{YM}^2} \delta(W) \sqrt{g} \frac{1}{2} Tr \left\{ \begin{aligned} & a^4 F_{mn} F^{mn} + 2a^2 (D_m A_\mu) (D^m A^\mu) \\ & - [A_\mu, A_\nu] [A^\mu, A^\nu] \end{aligned} \right\} \quad (4.10)$$

where all contractions in  $m$  labels are done by using the metric  $g_{mn}(\sigma)$  and in  $\mu$  labels by using the Minkowski metric  $\eta_{\mu\nu}$ .

Note that the ten fields  $A_\mu(\sigma)$  behave like scalar fields as functions of the two dimensional manifold  $\sigma^m$ . These are the 10 matrices of M(atrrix) theory that are covariant  $SO(9,1)$  vectors. Upon variation of the action with respect to the fields,  $W(\sigma), a(\sigma), g_{mn}(\sigma)$  as well  $A_\mu(\sigma)$ , we derive kinematic and dynamical equations for each field, as described earlier before Eq.(1.5). These can be solved easily, in particular by choosing the two dimensional basis labelled by  $(w, u)$  as it appears as a sub-basis in section (III A 1). The result of solving the kinematic equations is to produce the conformal shadow in which the fields shadow in which the fully reduced M(atrrix) theory fields  $A_\mu$  are now constants independent of  $\sigma^m$ , whose “dynamical equations are reproduced by the shadow action

$$L_{YM}^{shadow} = \frac{1}{4g_{YM}^2} \frac{1}{2} Tr([A_\mu, A_\nu] [A^\mu, A^\nu]) \quad (4.11)$$

This is the bosonic part of the supersymmetric M(atrix) Theory action in the (-1)-brane version [28]-[31].

The 0-brane version of [26]-[27] is derived similarly, by splitting the coordinates  $(10 + 2) \rightarrow (1 + 2) \oplus (9 + 0) \sim \sigma^m \oplus x^i$ , taking all the fields independent of  $x^i$ , and then following the same procedure as above.

Well before the action in Eq.(4.11) was interpreted as M(atrix) theory during 1996-99, this same action was proposed in 1990 as a bridge between string theory and large  $N$  gauge theory [25]. This was based on the observation that for  $N \rightarrow \infty$  one can substitute area preserving diffeomorphisms for  $SU(\infty)$ . In that case the infinite matrices  $(A_\mu)_i^j$  can be expressed in terms of string coordinates  $X_\mu(\xi^\alpha)$  on the worldsheet  $\xi^\alpha \equiv (\tau, \sigma)$ , matrix commutators are reproduced by Poisson brackets  $[A_\mu, A_\nu]_i^j \leftrightarrow \{X_\mu, X_\nu\}(\xi^\alpha) = \frac{\partial X_\mu}{\partial \tau} \frac{\partial X_\nu}{\partial \sigma} - \frac{\partial X_\nu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma}$ , and the trace of the infinite matrices is recovered by integration over the worldsheet  $Tr \leftrightarrow \int d^2\xi$ . Then the action in Eq.(4.11) is just proportional to  $\int d^2\xi \det(g)$  where the induced worldsheet metric is

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \eta_{\mu\nu}, \quad \det(g) = \{X_\mu, X_\nu\} \{X^\mu, X^\nu\}. \quad (4.12)$$

This is a gauge fixed version of the Nambu action

$$\int d^2\xi \det(-g) \leftrightarrow \int d^2\xi \det \sqrt{-g} \quad (4.13)$$

where the full diffeomorphism symmetry of the Nambu action has been gauge fixed to the subgroup of area preserving diffeomorphisms. The results of the present paper now show that all of this is recovered from the dimensional reduction of  $\text{SYM}_{10+2}^1$ .

We now turn to the fermionic terms. Starting from  $\text{SYM}_{10+2}^1$  in Eq.(1.2), after replacing the adjoint index  $a$  by matrices as above,

$$L_{SYM}^{fermi} = \frac{i}{2} \sqrt{-G} \delta(W(X)) \frac{1}{2} Tr \left[ \bar{\lambda} V \bar{D} \lambda + \bar{\lambda} \overleftarrow{D} V \lambda \right], \quad (4.14)$$

we follow the same procedure of dimensional reduction. We again have,  $\sqrt{-G} \delta(W(X)) = \sqrt{-g} a^{10} \delta(W(\sigma))$ . Also, because  $V^M$  and  $\partial_M$  are vanishing when  $M = \mu$ , we get

$$\begin{aligned} & \delta(W(X)) \sqrt{-G} \bar{\lambda} V \bar{D} \lambda \\ &= \delta(W(\sigma)) \sqrt{-g} a^{10} \bar{\lambda} (v_n \Gamma^n) \left\{ \bar{\Gamma}^m D_m \lambda + \bar{\Gamma}^\mu \left( \frac{1}{4} \omega_\mu^{ij} \Gamma_{ij} \lambda - i [A_\mu, \lambda] \right) \right\}. \end{aligned} \quad (4.15)$$

The spin connection  $\omega_M^{ij}$  that is compatible with the  $\text{Sp}(2, R)$  conditions in (1.5), namely  $E_M^i = D_M V^i$  has only the following non-zero components (with  $i = \hat{m} \oplus \hat{\mu}$  the tangent

indices)

$$\omega_M^{ij}(x, y) = \begin{array}{c|ccc} M \setminus ij & \hat{m}\hat{n} & \hat{m}\hat{\mu} & \hat{\mu}\hat{\nu} \\ \hline \omega_m^{ij} = & \omega_m^{\hat{m}\hat{n}} & 0 & 0 \\ \hline \omega_\mu^{ij} = & 0 & \omega_\mu^{\hat{m}\hat{\mu}} = \delta_\mu^{\hat{\mu}} \delta^{\hat{m}n} \partial_n \ln a & 0 \end{array} \quad (4.16)$$

The covariant derivative  $D_m$  includes  $\omega_m^{\hat{m}\hat{n}}$ , which is the standard spin connection constructed from a vielbein. The contribution from  $\omega_\mu^{\hat{m}\hat{\mu}}$  in Eq.(4.15) comes in the form

$$\frac{2}{4} \omega_\mu^{\hat{m}\hat{\mu}} \bar{\Gamma}^\mu \Gamma_{\hat{m}} \bar{\Gamma}_{\hat{\mu}} = \frac{1}{2} \delta_\mu^{\hat{\mu}} \delta^{\hat{m}n} \partial_n \ln a (-\Gamma_{\hat{m}} \Gamma^\mu \bar{\Gamma}_{\hat{\mu}}) = -\frac{10}{2} \bar{\Gamma}^n \partial_n \ln a \quad (4.17)$$

This is just right to absorb all dependence on the warp factor  $a(\sigma)$  into a rescaling of the fermion, as follows

$$(a^5 \bar{\lambda}) (v_n \Gamma^n) \{ \bar{\Gamma}^m D_m (\lambda a^5) + \bar{\Gamma}^\mu (-i [A_\mu, (\lambda a^5)]) \} \quad (4.18)$$

Thus, by defining a renormalized 32-spinor given by

$$\psi \equiv (\lambda a^5) \sqrt{4g_{YM}} \quad (4.19)$$

we manage to write the fermion action in the form

$$L_{SYM}^{fermi} = \frac{1}{4g_{YM}^2} \delta(W(\sigma)) \sqrt{-g} \frac{1}{2} Tr \left\{ \frac{1}{2} (i \bar{\psi} v \bar{\Gamma}^m D_m \psi + h.c.) + \bar{\psi} v \bar{\Gamma}^\mu [A_\mu, \psi] \right\} \quad (4.20)$$

Hence, the total reduced action for  $\text{SYM}_{10+2}^1$  is

$$S_{SYM}^{reduced} = \frac{1}{8g_{YM}^2} \int d^2\sigma \delta(W(\sigma)) \sqrt{-g} Tr \left\{ \begin{array}{l} -a^4 F_{mn} F^{mn} - 2a^2 (D_m A_\mu) (D^m A^\mu) \\ + \frac{1}{2} (i \bar{\psi} v \bar{\Gamma}^m D_m \psi + h.c.) \\ + [A_\mu, A_\nu] [A^\mu, A^\nu] + \bar{\psi} v \bar{\Gamma}^\mu [A_\mu, \psi] \end{array} \right\} \quad (4.21)$$

Note that here the field  $\lambda(\sigma)$  has 32 real components, but there is a kappa-type local symmetry, as in all 2T-physics actions that involve fermions [1], that eliminates half of the fermions by a gauge choice, thus really only 16 real fermion degrees of freedom are present. This is just the right content in M(atrrix) theory.

As outlined just before Eq.(4.11), solving the kinematic equations derived from this action produces the shadow which is recognized as the supersymmetrized M(atrrix) theory that generalizes Eq.(4.11), with matrices  $(A_\mu)_i^j$  and  $(\psi_+)_i^j$  that are independent of the two coordinates  $\sigma^m$

$$L_{SYM}^{shadow} = \frac{1}{8g_{YM}^2} Tr \{ [A_\mu, A_\nu] [A^\mu, A^\nu] + \bar{\psi}_+ \bar{\gamma}^\mu [A_\mu, \psi_+] \}. \quad (4.22)$$

Here  $\psi_+$  is the 16-component spinor of  $SO(9, 1)$  that corresponds to half of the 32-component  $\psi$ . Before choosing gauges or solving the kinematic equations, the 32 components  $\psi$  is a reminder and a link to 10+2 dimensions.

For large  $N$  this action (as well as its parent in Eq.(4.21)) may be rewritten in terms of Poisson brackets on a worldsheet [25] as in Eq.(4.12).

By using similar methods, other versions of M(atrix) theory that relate to 0-branes, 1-branes, and more generally p-branes [26]-[27] can be derived directly from the action of  $SYM_{10+2}^1$  in Eq.(1.2) by various dimensional reductions or compactifications that parallel those in [26]-[27].

## V. CLOSING COMMENTS

Having established that the conventional 1T-physics methods miss systematically a vast amount of information even in simple classical or quantum mechanics (see recent summary [9] and the introduction in [10]) it is reasonable to expect that progress in fundamental physics, in particular the quest for the fundamental principles, would benefit from the methods of 2T-physics. It is with this in mind that we have embarked on constructing the higher dimensional 2T theories that connect to well known and cherished theories in 1T-physics. In this paper we have discussed the first such theory in 10+2 dimensions, a number of dimensions that was not reached before, and have shown that it is the source, and unifying factor, of well known lower dimensional theories.

The process of derivation is a combination of dimensional reduction and extracting a shadow of 2T-physics by solving the kinematic equations that follow from the 2T action. The kinematic equations amount to imposing the gauge symmetry requirements of  $Sp(2, R)$  in phase space, as summarized recently in [9][10]. In principle there are many other types of shadows and compactifications derivable from 2T-field theory that can lead to other dual versions of each of the theories discussed here, as sketched in Fig.1. The additional shadows produced by 2T field theory have so far been little explored in the context of field theory [4][5] although they are much better developed in the context of classical or quantum particle mechanics [9].

By using the web of connections that we discussed here, and those that can be further derived, one can in principle establish a web of dualities or connections among various 1T-

theories that were not suspected before. This additional predicted information, which can be verified in 1T-physics, is related to the extra dimensions as is already captured by the same unifying theory in 10+2 dimensions. Hence, studying directly the theory in 10+2 dimensions (for example as in [4][5][24][23]) can yield many benefits and predictions for the lower dimensional theories. In addition to the deeper implications that our program has about the meaning of space-time, exploring the hidden symmetries and dualities related to the 10+2 dimensional parent  $\text{SYM}_{10+2}^1$  theory is expected to yield many practical side benefits, including new computational techniques that could clarify or supplement those already used in  $\text{SYM}_{3+1}^4$  and in M(atrix) theory.

The path of research pursued in the current paper is expected to lead to supergravity in 10+2 and 11+2 dimensions and eventually to a 2T approach to M-theory and its dualities. This should provide a dynamical and gauge symmetry basis for F-theory [33] and S-theory [34] from deeper phase space gauge symmetry principles [9][35] which require higher spacetime with two times.

## Appendix A: Conformally flat shadow spacetimes in $d$ dimensions from flat $d + 2$ spacetime

The topic of this appendix was part of the discussion in [4][5] on the shadows of 2T field theory in flat spacetime. But in this appendix we present a more systematic approach for the conformal shadow, including the expansion in powers of  $w$  that was not covered in [4][5].

Consider the line element in flat spacetime in  $d + 2$  dimensions parametrized as

$$ds_{d+2}^2 = dX^i dX^j \eta_{ij} = - \left( dX^{0'} \right)^2 + \left( dX^{1'} \right)^2 + dX^\alpha dX^a \eta_{ab} \quad (\text{A1})$$

$$= -2dX^{+'} dX^{-'} + dX^\alpha dX^a \eta_{ab} , \quad (\text{A2})$$

where  $\eta_{ij}$  is the flat metric with  $\text{SO}(d, 2)$  symmetry and  $\eta_{ab}$  is the Minkowski metric with  $\text{SO}(d - 1, 1)$  symmetry. We parametrize these flat Cartesian coordinates  $X^i$ , with  $i = (\pm', a)$  labeling the flat basis, in terms of curvilinear coordinates  $X^M = (w, u, x^\mu)^M$ , where  $M$  labels the curvilinear basis (hence compared to the curved basis in the text), as follows

$$X^{+'} = \frac{X^{0'} + X^{1'}}{\sqrt{2}} = \pm e^{-2\Sigma}, \quad X^a = e^{-2\Sigma} q^a, \quad (\text{A3})$$

$$X^{-'} = \frac{X^{0'} - X^{1'}}{\sqrt{2}} = \pm e^{-2\Sigma} \frac{q^2}{2} \mp e^{2\Sigma} \frac{w}{2}, \quad (\text{A4})$$

where  $\Sigma$  and  $q^a$  are arbitrary functions of the curvilinear coordinates  $(w, u, x)$ . The point of this parametrization is that computing  $X^2 = X^i X^j \eta_{ij}$  we find  $X^2 = w$ . That is,  $X^2$  as computed with the flat Cartesian coordinates  $X^i$  coincides with the curvilinear coordinate  $w$ . After computing  $dX^i$  and inserting in  $ds_{d+2}^2 = dX^i dX^j \eta_{ij}$  the flat metric above takes the form

$$ds_{d+2}^2 = -2dw (d\Sigma) - 4w (d\Sigma)^2 + e^{-4\Sigma} (dq)^2. \quad (\text{A5})$$

We will take the following specialized form for  $\Sigma(w, u, x)$  and  $q^a(w, u, x)$

$$\Sigma(w, u, x) = u + \frac{1}{2} \sigma(x, w e^{4u}), \quad q^a(w, u, x) = q^a(x, w e^{4u}). \quad (\text{A6})$$

This form is motivated by previous work [4][5][6][7] which shows the relevance of the com-

bination of coordinates  $we^{4u} \equiv z$ . This gives

$$ds_{d+2}^2 = \left\{ \begin{array}{l} -(dw)^2 [\sigma' (1 + we^{4u} \sigma') - e^{-2\sigma} (q')^2] e^{4u} \\ + (du)^2 [-4w + 16 (we^{4u})^2 [\sigma' (1 + we^{4u} \sigma') - e^{-2\sigma} (q')^2]] \\ + dx^\mu dx^\nu [-(we^{4u}) \partial_\mu \sigma \partial_\nu \sigma + e^{-2\sigma} \partial_\mu q \cdot \partial_\nu q] e^{-4u} \\ + 2dwdu [-1 - 4z (\sigma' (1 + we^{4u} \sigma') - e^{-2\sigma} (q')^2)] \\ + 2dw dx^\mu [-\left(\frac{1}{2} + we^{4u} \sigma'\right) \partial_\mu \sigma + e^{-2\sigma} \partial_\mu q \cdot q'] \\ + 2dudx^\mu [-\left(\frac{1}{2} + we^{4u} \sigma'\right) \partial_\mu \sigma + e^{-2\sigma} \partial_\mu q \cdot q'] 4w \end{array} \right\}.$$

where  $\sigma'$  and  $q'_a$  are defined as the *total* derivatives with respect to the variable  $z \equiv we^{4u}$

$$q'_a \equiv \frac{dq^a(x, z)}{dz}, \quad \sigma' \equiv \frac{d\sigma(q^a(x, z), z)}{dz} = \frac{\partial \sigma}{\partial z} + \frac{\partial \sigma}{\partial q^a} q'_a. \quad (\text{A7})$$

It was argued in [6][7] that a general metric in 2T-gravity (therefore in particular the flat case in this Appendix) can be brought to the following standard gauge fixed form which is appropriate for the conformal shadow and its prolongations [7] at any  $w$

$$ds^2 = -2dwdu - 4w (du)^2 + e^{-4u} g_{\mu\nu}(x, we^{4u}) dx^\mu dx^\nu. \quad (\text{A8})$$

If the above expression for  $ds_{d+2}^2$  is to agree with this gauge fixed form we must put further constraints on  $\sigma, q^a$  as follows

$$e^{-2\sigma} (q')^2 = \sigma' (1 + we^{4u} \sigma'), \quad (\text{A9})$$

$$2\partial_\mu q \cdot q' = (1 + 2we^{4u} \sigma') e^{2\sigma} \partial_\mu \sigma. \quad (\text{A10})$$

These equations are solved uniquely by the following expressions for  $\sigma', q'_a$  (using the chain rule  $\partial_\mu \sigma = \frac{\partial q^a}{\partial x^\mu} \frac{\partial \sigma}{\partial q^a}$ )

$$q'_a = \frac{e^{2\sigma} \partial_a \sigma}{2\sqrt{1 - ze^{2\sigma} (\partial_a \sigma)^2}}, \quad (\text{A11})$$

$$\sigma' = \frac{e^{2\sigma} (\partial_a \sigma)^2}{2\sqrt{1 - ze^{2\sigma} (\partial_a \sigma)^2} \left(1 + \sqrt{1 - ze^{2\sigma} (\partial_a \sigma)^2}\right)}. \quad (\text{A12})$$

where  $\partial_a \sigma \equiv \frac{\partial \sigma}{\partial q^a}$ , and  $(\partial_a \sigma)^2 \equiv \eta^{ab} \partial_a \sigma \partial_b \sigma$ . We see that (A12) is a partial differential equation for  $\sigma(q^a, z)$  as a function of  $d+1$  coordinates  $q^a, z$ . Once  $\sigma(q^a, z)$  is determined by solving this equation, we can find  $q_a(x, z)$  by integrating the first equation with respect to  $z$

$$q^a(x, z) = q_0^a(x) + \int_0^z dz' \frac{e^{2\sigma} \partial_b \sigma \eta^{ab}}{2\sqrt{1 - z'e^{2\sigma} (\partial_a \sigma)^2}}. \quad (\text{A13})$$

where  $q_0^a(x)$  is completely arbitrary.

There remains solving the  $\sigma$ -equation (A12). It is useful to do this by expanding both  $q^a(x, z)$  and  $\sigma(q^a(x, z), z)$  in powers of  $z (= we^{4u})$  since after all we are only interested in the first few powers in  $z$  on account of the delta function  $\delta(w)$  in the action (1.1). So, we define the expansion

$$q^a(x, z) = q_0^a(x) + zq_1^a(x) + \frac{z^2}{2}q_2^a(x) + \dots \quad (\text{A14})$$

$$\sigma(q^a(x, z), z) = \sigma_0(q_0(x)) + z\sigma_1(q_0(x)) + \frac{z^2}{2}\sigma_2(q_0(x)) + \dots \quad (\text{A15})$$

By inserting these back into the  $\sigma$ -equation (A12) and noting that we can use  $\partial_a \sigma = \frac{\partial \sigma}{\partial q^a} = \frac{\partial \sigma}{\partial q_0^a}$ , we easily obtain an explicit solution for  $\sigma_1, \sigma_2, \dots$ , and  $q_1^a, q_2^a, \dots$  in terms of the arbitrary  $d+1$  functions of spacetime  $q_0^a(x), \sigma_0(q_0(x))$ . The result looks as follows up to  $O(z^2)$

$$\begin{aligned} q^a(x, z) &= q_0^a(x) + ze^{2\sigma_0} \frac{\partial \sigma_0(q_0)}{2\partial q_0^a} + \dots \\ \sigma(x, z) &= \sigma_0(q_0) + ze^{2\sigma_0} \left( \frac{\partial \sigma_0(q_0)}{2\partial q_0^a} \right)^2 + \dots \\ q_0^a(x), \sigma_0(q_0(x)) &\text{ are arbitrary.} \end{aligned} \quad (\text{A16})$$

It is evident that the coefficients of all higher powers in  $z$  in both  $\sigma(x, z)$  and  $q^a(x, z)$  are completely fixed by the  $d+1$  arbitrary functions  $q_0^a(x), \sigma_0(q_0(x)) = \sigma_0(x)$ .

With this result we now analyze again the line element which now has the standard gauge fixed form (for the conformal shadow)

$$ds^2 = -2dwdu - 4w(du)^2 + e^{-4u}g_{\mu\nu}(x, we^{4u})dx^\mu dx^\nu, \quad (\text{A17})$$

and find that  $g_{\mu\nu}(x, z)$  is given by

$$g_{\mu\nu}(x, z) = e^{-2\sigma} \partial_\mu q^a \partial_\nu q^b \eta_{ab} - z \partial_\mu \sigma \partial_\nu \sigma \quad (\text{A18})$$

$$= g_{\mu\nu}^{(0)}(x) + zg_{\mu\nu}^{(1)}(x) + \frac{z^2}{2}g_{\mu\nu}^{(2)}(x) + \dots \quad (\text{A19})$$

By inserting our solutions for  $q^a(x, z)$  and  $\sigma(x, z)$  we compute  $g_{\mu\nu}^{(0)}, g_{\mu\nu}^{(1)}, g_{\mu\nu}^{(2)}, \dots$  as follows

$$g_{\mu\nu}^{(0)}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab}, \quad \text{where } e_\mu^a(x) = e^{-\sigma_0(x)} \frac{\partial q_0^a(x)}{\partial x^\mu}, \quad (\text{A20})$$

$$g_{\mu\nu}^{(1)}(x) = -\frac{1}{2}(\partial_a \sigma_0)^2 \partial_\mu q_0 \cdot \partial_\nu q_0 + \partial_\mu \sigma_0 \partial_\nu \sigma_0 + (\partial_\mu q_0^b \partial_\nu q_0^a) \partial_b \partial_a \sigma_0, \quad (\text{A21})$$

$$g_{\mu\nu}^{(2)}(x) = \dots \quad (\text{A22})$$



In the expression for  $g_{\mu\nu}^{(1)}(x)$ , assuming that  $\sigma_0(q_0(x)) = \sigma_0(x)$  is chosen as a function of  $x^\mu$ , we can evaluate the derivatives  $\partial_a \sigma_0$  by using the chain rule

$$\partial_a \sigma_0(x(q_0)) = \frac{\partial x^\mu}{\partial q_0^a} \partial_\mu \sigma_0(x) = e^{-\sigma_0(x)} e_a^\mu(x) \partial_\mu \sigma_0(x), \quad (\text{A23})$$

where  $e_a^\mu(x)$  is the inverse of the vielbein defined in (A20). So, the simple rule is  $\partial_a(e^{\sigma_0}) = e_a^\mu(x) \partial_\mu \sigma_0(x)$ .

The lowest component  $g_{\mu\nu}^{(0)}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab}$  alone determines the geometric properties of the shadow in  $d$  dimensions [7], and from the form of the vielbein  $e_\mu^a(x) = e^{-\sigma_0(x)} \frac{\partial q_0^a(x)}{\partial x^\mu}$  we see that the spacetime of the shadow is a conformally flat spacetime.

The higher components of the metric  $g_{\mu\nu}^{(1)}(x), g_{\mu\nu}^{(2)}(x), \dots$  determine the geometric properties of the *prolongations* of the shadow as discussed in [7], but these do not interfere with the self consistent 1-time physics of the shadow in the spacetime given by  $g_{\mu\nu}^{(0)}(x)$  [7].

## Appendix B: More general solution of the SUSY condition

In this appendix we find a more general solution of the SUSY condition (1.7) which is regarded as a constraint on the SUSY parameter  $\varepsilon_A$

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{PQN} \Gamma^M \varepsilon)_A V_N \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{PQN} D_M \varepsilon)_A V_N \right]_{W=0} = \left[ V^P U_A^Q - V^Q U_A^P \right]_{W=0}. \quad (\text{B1})$$

Recall that the  $U_A^P$  are arbitrary. In the  $X^M = (w, u, x^\mu)^M$  basis, appropriate for the conformal shadow, we use the results in Eqs.(3.9-3.17), in particular  $V_N = (\frac{1}{2}, 0, 0)_N$  and  $V^N = (2w, -\frac{1}{2}, 0)^N$  and then set  $W(X) = w = 0$  to simplify this expression

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{PQw} \Gamma^M \varepsilon)_A \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{PQw} D_M \varepsilon)_A \right]_{w=0} = - \left( \delta_u^P U_A^Q - \delta_u^Q U_A^P \right)_{w=0}. \quad (\text{B2})$$

We will also use the expressions for the spin connection  $\omega_M^{ij}$  given in (3.14,3.15) at  $w = 0$  (after derivatives  $\partial_w$  are taken). Next we specialize the antisymmetric indices  $[PQ]$  to examine systematically the various tensor components of the SUSY condition as follows.

For  $PQ = wu$  we have

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{wuw} \Gamma^M \varepsilon)_A \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{wuw} D_M \varepsilon)_A \right]_{w=0} = - (\delta_u^w V^w U_A^u - \delta_u^u U_A^w)_{w=0}. \quad (\text{B3})$$

Noting that at  $w = 0$ , we get  $\bar{\Gamma}^{wuw} = \Gamma^{wuw} = \delta_u^w = 0$  and  $\delta_u^u = 1$ , this equation determines  $U_A^w$

$$U_A^w = 0. \quad (\text{B4})$$

Next we take  $PQ = w\lambda$

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{w\lambda w} \Gamma^M \varepsilon)_A \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{w\lambda w} D_M \varepsilon)_A = -(\delta_u^w U_A^\lambda - \delta_u^\lambda U_A^w) \right]_{w=0}. \quad (\text{B5})$$

Every term on both sides of this equation vanishes since  $\Gamma^{w\lambda w} = \delta_u^w = \delta_u^\lambda = 0$ , so this is an identity. Next we take  $PQ = u\lambda$

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{u\lambda w} \Gamma^M \varepsilon)_A \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{u\lambda w} D_M \varepsilon)_A = -(\delta_u^u U_A^\lambda - \delta_u^\lambda U_A^u) \right]_{w=0}.$$

We use  $\bar{\Gamma}^{u\lambda w} = \bar{\Gamma}^{wu} \bar{\Gamma}^\lambda$  and  $\Gamma^{u\lambda w} = \Gamma^{wu} \Gamma^\lambda$  to determine  $U_A^\lambda$  in the form

$$U^\lambda = -\bar{\Gamma}^{wu} \left[ -\frac{d-4}{d-2} \bar{\Gamma}^\lambda \Gamma^M \varepsilon \partial_M \ln \Omega + (\tilde{\Gamma}^M \Gamma^\lambda D_M \varepsilon) \right]_{w=0} \quad (\text{B6})$$

where  $\tilde{\Gamma}^M = \bar{\Gamma}^{wu} \bar{\Gamma}^M \bar{\Gamma}^{wu} = (-\Gamma^w, -\Gamma^u, \Gamma^\mu)^M$ . In this expression we are supposed to insert  $\Omega(X) = e^{(d-2)u} \hat{\Omega}(x, we^{4u})$  and  $\varepsilon(X) = \exp(u\Gamma^{+-'}) \hat{\varepsilon}(x, we^{4u})$  as determined (3.17) and set  $w = 0$ .

So far there has been no conditions on  $\hat{\varepsilon}_A$ , but the next case of  $PQ = \nu\lambda$  produces conditions on  $\hat{\varepsilon}_A$  as follows

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{\nu\lambda w} \Gamma^M \varepsilon) \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{\nu\lambda w} D_M \varepsilon) = -(\delta_u^\nu U_A^\lambda - \delta_u^\lambda U_A^\nu) \right]_{w=0}.$$

After using  $\bar{\Gamma}^{\nu\lambda w} = \bar{\Gamma}^{\nu\lambda} \bar{\Gamma}^w$  and noting that the right hand side vanishes on account of  $\delta_u^\nu = 0$ , we get

$$\left[ -\frac{d-4}{d-2} (\bar{\Gamma}^{\nu\lambda} \bar{\Gamma}^w \Gamma^M \varepsilon) \partial_M \ln \Omega + (\bar{\Gamma}^M \Gamma^{\nu\lambda} \Gamma^w D_M \varepsilon) \right]_{w=0} = 0. \quad (\text{B7})$$

In this expression both  $M = w$  terms drop because  $[\bar{\Gamma}^w \Gamma^w]_{w=0} = [G^{ww}]_{w=0} = 0$ . Furthermore, the term  $D_u \varepsilon$  drops because  $[D_u \varepsilon]_{w=0} = 0$  as in (3.17), and we can set  $\partial_u \ln \Omega = d-2$  at  $w = 0$  on account of (3.17). The result has the form

$$0 = \left\{ \bar{\Gamma}^w \Gamma^{\nu\lambda} \left[ -(d-4) \Gamma^u \varepsilon - \frac{d-4}{d-2} (\partial_\mu \ln \Omega) \Gamma^\mu \varepsilon \right] - \bar{\Gamma}^w \Gamma^\mu \bar{\Gamma}^{\nu\lambda} D_\mu \varepsilon \right\}_{w=0}. \quad (\text{B8})$$

As an example consider  $d+2 = 12$  (for the other cases  $d+2 = 5, 6, 8$  the discussion is similar, by changing only the size of the spinor). Since in 12-dimensions  $[\bar{\Gamma}^w]_{w=0}$  is a  $32 \times 32$  matrix proportional to  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  where each entry is a  $16 \times 16$  matrix, this equation amounts to 16 equations imposed on the 32 components of  $\varepsilon_A$ . Taking into account (3.17) we write the 32 component  $\varepsilon_A(X)$  in terms of two 16-component pieces  $\varepsilon_1(x), \varepsilon_2(x)$  at  $w = 0$

$$[\varepsilon_A(X)]_{w=0} = \begin{pmatrix} e^{-u} \varepsilon_1(x) \\ e^{+u} \varepsilon_2(x) \end{pmatrix} \quad (\text{B9})$$

Then the 16 equations above take the form<sup>11</sup>

$$0 = -(d-4) \left( -i\sqrt{2} \right) \bar{\gamma}^{\nu\lambda} \varepsilon_2 - \partial_\mu \ln \phi^{\frac{d-4}{d-2}} \bar{\gamma}^{\nu\lambda} \bar{\gamma}^\mu \varepsilon_1 - \bar{\gamma}^\mu \gamma^{\nu\lambda} \left( D_\mu \varepsilon_1 + i\sqrt{2} \gamma_\mu \varepsilon_2 \right),$$

where in the expression for  $D_\mu \varepsilon_1$  only the usual 1T form of the spin connection  $\omega_\mu^{ab}$  appears. Now we use  $\bar{\gamma}^\mu \gamma^{\nu\lambda} \gamma_\mu = (d-4) \gamma^{\nu\lambda}$  and notice that  $\varepsilon_2$  drops out of this equation, so the constraint on  $\varepsilon$  simplifies to a constraint only on  $\varepsilon_1(x)$

$$-\bar{\gamma}^{\nu\lambda} \bar{\gamma}^\mu \varepsilon_1 \left( \partial_\mu \ln \phi^{\frac{d-4}{d-2}} \right) - \bar{\gamma}^\mu \gamma^{\nu\lambda} D_\mu \varepsilon_1 = 0, \quad \varepsilon_2(x) = \text{arbitrary.} \quad (\text{B10})$$

The equation for  $\varepsilon_1$  can be manipulated by contracting with  $\bar{\gamma}_{\nu\lambda}$  and using

$$\bar{\gamma}_{\nu\lambda} \bar{\gamma}^{\nu\lambda} = -d(d-1), \quad \bar{\gamma}_{\nu\lambda} \bar{\gamma}^\mu \gamma^{\nu\lambda} = -(d-1)(d-4) \bar{\gamma}^\mu, \quad (\text{B11})$$

to extract the following expression

$$\left( \partial_\mu \ln \phi^{\frac{d}{d-2}} \bar{\gamma}^\mu \varepsilon_1 + \bar{\gamma}^\mu D_\mu \varepsilon_1 \right) (d-1)(d-4) = 0. \quad (\text{B12})$$

After some manipulation of gamma matrices Eq.(B10) is simplified to the following form (to verify use  $\bar{\gamma}^\mu \gamma^{\nu\lambda} \gamma_\mu = (d-4) \bar{\gamma}^{\nu\lambda}$ )

$$D_\mu \varepsilon_1 = \frac{1}{d} \gamma_\mu (\bar{\gamma} \cdot D \varepsilon_1) \quad \text{and} \quad (d-4) \bar{\gamma}^\mu D_\mu \left( \phi^{\frac{d}{d-2}} \varepsilon_1 \right) = 0, \quad \varepsilon_2(x) = \text{arbitrary.} \quad (\text{B13})$$

The second equation is trivially satisfied if  $d+2=6$  so it is a restriction on the  $\text{SO}(d,1)$  spinor  $\varepsilon_1$  only when  $d+2=5, 8, 12$ .

Now consider flat space as an example, with  $\partial_\mu \phi = 0$ , and  $\omega_\mu^{ab} = 0$ . The solutions of these equations are

$$\varepsilon_1 = \varepsilon_1^0 + x \cdot \gamma \tilde{\varepsilon}_1^0, \quad \text{for } d+2=6, \quad \text{with } \varepsilon_1^0, \tilde{\varepsilon}_1^0 \text{ constant spinors of } \text{SO}(3,1). \quad (\text{B14})$$

$$\varepsilon_1 = \varepsilon_1^0, \quad \text{for } d+2=5, 8, 12, \quad \text{with } \varepsilon_1^0, \text{ constant spinor of } \text{SO}(d,1). \quad (\text{B15})$$

Note that in the flat case for  $d+2=6$ , the complex  $\text{SO}(3,1)$  spinors  $\varepsilon_1^0, \tilde{\varepsilon}_1^0$  correspond to supersymmetry and superconformal transformations respectively, and their closure gives the superalgebra  $\text{SU}(2,2|1)$ . On the other hand, for  $d+2=12$  the  $\text{SO}(9,1)$  spinor  $\varepsilon_1^0$  is real and contains only 16 components, so this case has only 16 supersymmetries, but not

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<sup>11</sup> In arriving at this expression we have used the 2T form of the spin connection in (3.15) to evaluate  $(D_\mu \varepsilon)_A = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \bar{\Gamma}_{ab} + \frac{1}{2} \omega_\mu^{+'b} \bar{\Gamma}_b \Gamma^{+'} + \frac{1}{2} \omega_\mu^{-'b} \bar{\Gamma}_b \Gamma^{+'} \right) \varepsilon_A$ ,

superconformal symmetry. In a more general curved space  $\varepsilon_1(x)$  may depend on  $x^\mu$  even when  $d+2 = 5, 8, 12$ , and may thus contain more than one constant spinor of  $\text{SO}(d, 1)$ , thus possibly having more than 16 supersymmetries.

The number of supersymmetries may be determined also by analyzing the number of conserved currents associated with constant spinor parameters. The conserved current discussed in the text is

$$\bar{\varepsilon} J^M = \delta(W) \sqrt{G} \Omega^{\frac{d-4}{d-2}} F_{PQ}^a V_N \bar{\varepsilon} (\Gamma^{PQN} \bar{\Gamma}^M) \lambda^a. \quad (\text{B16})$$

Here in general  $\bar{\varepsilon}(X)$  depends on the  $X^M$  in  $d+2$  dimensions, while this  $\bar{\varepsilon}(X)$  satisfies the SUSY condition (1.7). Let's write every component of  $\bar{\varepsilon} J^M$  in the  $(w, u, x)$  basis, and in the gauge in which

$$\lambda = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} e^{(d-1)u}, \quad F_{wu} = F_{w\mu} = F_{u\mu} = 0, \quad \Omega = \phi e^{(d-2)u} \quad (\text{B17})$$

$$\sqrt{G} = e^{-2du} \sqrt{-g}, \quad \Gamma^w \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^u \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (\text{B18})$$

all evaluated at  $w = 0$ . We get

$$\bar{\varepsilon} J^w = \frac{1}{2} \delta(w) e^{-5u} \sqrt{-g} \phi^{\frac{d-4}{d-2}} F_{Pq}^a \bar{\varepsilon} (\Gamma^{Pq} \Gamma^w \bar{\Gamma}^w) \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = 0, \quad (\text{B19})$$

$$\bar{\varepsilon} J^u = \frac{1}{2} \delta(w) e^{-5u} \sqrt{-g} \phi^{\frac{d-4}{d-2}} F_{Pq}^a \bar{\varepsilon} (\Gamma^{Pq} \Gamma^u \bar{\Gamma}^u) \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = 0, \quad (\text{B20})$$

$$\bar{\varepsilon} J^\mu = \frac{1}{2} \delta(w) e^{-5u} \sqrt{-g} \phi^{\frac{d-4}{d-2}} F_{Pq}^a \bar{\varepsilon} (\Gamma^{Pq} \Gamma^\mu \bar{\Gamma}^\mu) \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} \quad (\text{B21})$$

$$\sim -\frac{1}{2} \delta(w) e^{-5u} \sqrt{-g} \phi^{\frac{d-4}{d-2}} F_{Pq}^a (\bar{\varepsilon}_2 \bar{\varepsilon}_1) (\Gamma^{Pq} \bar{\Gamma}^\mu) \begin{pmatrix} 0 \\ \lambda_1 \end{pmatrix} \quad (\text{B22})$$

$$= -\frac{1}{2} \delta(w) e^{-5u} \sqrt{-g} \phi^{\frac{d-4}{d-2}} F_{Pq}^a \bar{\varepsilon}_1 (-\bar{\gamma}^{Pq} \gamma^\mu) \lambda_1 \quad (\text{B23})$$

So  $\varepsilon_2$  does not contribute at all in the conformal shadow.  $\varepsilon_1$  is a real spinor of  $\text{SO}(9, 1)$  so it has 16 components, implying 16 conserved SUSY currents or 16 supersymmetries if  $\varepsilon_1$  is just a constant spinor. Among remaining questions in 12-dimensions is whether there are spacetimes with nontrivial  $\bar{\varepsilon}_1(x)$  that contain more than 16 constant spinor components (for example, an analog of Eq.(B14) in 4-dimensions), thus implying more than 16

supersymmetries? The fact that the compactified version of the 12-dimensional theory  $(10+2) \rightarrow (4+2)$ , shown in Fig.1, is symmetric under  $SU(2,2|4)$  and contains 32 supersymmetries, is an indication that there may be non-trivial backgrounds  $W, \Omega, G_{MN}$  in which there are at least 32 supersymmetries, but we have not identified them in this paper.

### Appendix C: $SO(10,2)$ spinors in $SU(2,2) \times SU(4)$ basis

We label the  $SO(10,2)$  real spinor **32** in the  $SU(2,2) \times SU(4)$  basis as  $\psi_\alpha^r$ , which is a complex  $(4, \bar{4})$ . The spinor labels  $\rho, r$  in this section should not be confused with the vector labels  $\alpha, m$  used in the previous section. Its conjugate  $\bar{\psi}$  will be labelled as  $\bar{\psi}_r^\rho$ , which is a  $(\bar{4}, 4)$  and is constructed by taking Hermitian conjugation and multiplying by the  $SU(2,2)$  metric in spinor space  $\eta^{\dot{\rho}\sigma}$  (see appendix of [2])

$$\bar{\psi} = \psi^\dagger \eta; \quad \bar{\psi}_r^\rho = (\psi^\dagger)_{r\dot{\sigma}} \eta^{\dot{\sigma}\rho}. \quad (C1)$$

The charge conjugate spinor  $\psi^c$  is given by taking the transpose of  $\bar{\psi}$  and multiplying by the charge conjugation matrix

$$\psi^c = C \bar{\psi}^T; \quad (\psi^c)_{\dot{\rho}r} = C_{\dot{\rho}\sigma} (\bar{\psi}^T)^\sigma_r = C_{\dot{\rho}\sigma} (\eta^T)^{\sigma\dot{\kappa}} (\psi^*)_{\dot{\kappa}r} = \tilde{C}_{\dot{\rho}\dot{\kappa}} (\psi^*)_{\dot{\kappa}r} \quad (C2)$$

We define a pseudoreal spinor basis of  $SO(10,2)$  that has 32 real components (constructed from the 16 complex components of  $\psi$  or  $\psi^c$ ) as follows (here we suppress the Yang-Mills group adjoint representation label)

$$\psi_A = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\rho^r \\ (\psi^c)_{\dot{\rho}r} \end{pmatrix} \sim 32 = \begin{pmatrix} (4, \bar{4}) \\ (\bar{4}, 4) \end{pmatrix} \quad (C3)$$

The normalization of  $1/\sqrt{2}$  is to insure the correct normalization of kinetic terms in terms of  $\psi$ .

The  $SO(10,2)$  transformation laws of the 32-spinor  $\psi_A$

$$\delta_\omega \psi_A = -\frac{1}{4} \omega^{MN} (\Gamma_{MN})_A^B \psi_B, \quad (C4)$$

can be rewritten in the  $SO(4,2) \times SO(6) = SU(2,2) \times SU(4)$  basis as the following of  $SO(10,2)$  transformation laws of  $\psi_\rho^r \sim (4, \bar{4})$

$$(\delta_\omega \psi)_\rho^r = -\frac{1}{4} \omega^{mn} (\gamma_{mn} \psi)_\rho^r + \frac{1}{4} \omega^{IJ} (\psi \gamma_{IJ})_\rho^r + \frac{1}{2} \omega^{mI} (\gamma_m \psi^c \gamma_I)_\rho^r \quad (C5)$$

$$= -\frac{1}{4} \omega^{mn} (\gamma_{mn})_\rho^\sigma \psi_\sigma^r + \frac{1}{4} \omega^{IJ} \psi_\rho^s (\gamma_{IJ})_s^r + \frac{1}{2} \omega^{mI} (\gamma_m)_\rho^{\dot{\sigma}} (\psi^c)_{\dot{\sigma}s} (\bar{\gamma}_I)^{sr} \quad (C6)$$

Here all small  $\gamma$ 's are  $4 \times 4$  matrices expressed in the spinor bases of  $SU(2, 2)$  or  $SU(4)$ . From these we compute the transformation laws for the charge conjugate spinor  $\psi^c \sim (\bar{4}, 4)$  as

$$(\delta_\omega \psi^c)_{\dot{p}r} = -\frac{1}{4}\omega^{mn}(\bar{\gamma}_{mn}\psi^c)_{\dot{p}r} + \frac{1}{4}\omega^{IJ}(\psi^c\bar{\gamma}_{IJ})_{\dot{p}r} + \frac{1}{2}\omega^{mI}(\bar{\gamma}_m\psi\gamma_I)_{\dot{p}r} \quad (C7)$$

$$= -\frac{1}{4}\omega^{mn}(\bar{\gamma}_{mn})_{\dot{p}}^{\dot{s}}(\psi^c)_{\dot{s}r} + \frac{1}{4}\omega^{IJ}(\psi^c)_{\dot{p}s}(\bar{\gamma}_{IJ})^s_r + \frac{1}{2}\omega^{mI}(\bar{\gamma}_m)_{\dot{p}}^{\dot{s}}\psi_\sigma^s(\gamma_I)_{sr} \quad (C8)$$

These are consistent with  $\delta_\omega \psi^c = C\overline{(\delta_\omega \psi)}^T = C\eta^T(\delta_\omega \psi)^*$  since

$$C\eta^T(\gamma_{mn})^*(\eta^T)^{-1}C^{-1} = \bar{\gamma}_{mn}, \quad C\eta^T(\gamma_m)^*(\eta^T)^{-1}C^{-1} = -\bar{\gamma}_m \quad (C9)$$

$$(\gamma_{IJ})^* = \bar{\gamma}_{IJ}, \quad (\bar{\gamma}_I)^* = -\gamma_I; \text{ also } (\gamma_I)_{rs}, (\bar{\gamma}_I)^{rs} \text{ are antisymmetric} \quad (C10)$$

The last line also implies

$$(\bar{\gamma}_I)^\dagger = \gamma_I, \text{ and } (\gamma_{IJ})^\dagger = -\gamma_{IJ}, \quad (\bar{\gamma}_{IJ})^\dagger = -\bar{\gamma}_{IJ} \quad (C11)$$

which is consistent with Hermitian  $SU(4)$  generators  $\frac{i}{2}\gamma_{IJ}$  and  $\frac{i}{2}\bar{\gamma}_{IJ}$  in the  $\mathbf{4}$  and  $\bar{\mathbf{4}}$  representations respectively. The explicit matrix form of the antisymmetric  $SO(6)$  gamma matrices  $(\gamma_I)_{rs}, (\bar{\gamma}_I)^{rs}$  can be taken as

$$(\gamma_I)_{rs} = ((\sigma_2 \times i\sigma_2 \vec{\sigma}), (\sigma_2 \vec{\sigma} \times \sigma_2)), \quad (\text{note } i\sigma_2 \vec{\sigma} = (\sigma_1, i, -\sigma_3)) \quad (C12)$$

$$(\bar{\gamma}_I)^{rs} = ((\sigma_2 \times i\sigma_2 \vec{\sigma}^*), (-\sigma_2 \vec{\sigma}^* \times \sigma_2)), \quad (\text{note } (-\sigma_2 \vec{\sigma}^*) = (i\sigma_3, 1, -i\sigma_1)) \quad (C13)$$

These satisfy the Clifford algebra property of  $SO(6)$  gamma matrices

$$(\gamma_I \bar{\gamma}_J + \gamma_J \bar{\gamma}_I)_r^s = 2\delta_{IJ}\delta_r^s. \quad (C14)$$

Some further property is that for each  $I$  these satisfy

$$(\bar{\gamma}_I)^{rs} = \frac{1}{2}\varepsilon^{rsuv}(\gamma_I)_{uv}. \quad (C15)$$

The  $SO(10, 2)$  transformation laws of the 32-component spinor  $\delta_\omega \psi_A = -\frac{1}{4}\omega^{mn}(\gamma_{mn})_A^B \psi_B$  can now be written in the form of  $32 \times 32$  matrices

$$\delta_\omega \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} = -\frac{1}{4}\omega^{MN}(\Gamma_{MN}) \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} \quad (C16)$$

$$\omega^{MN}(\Gamma_{MN}) = \begin{pmatrix} \omega^{mn}(\gamma_{mn} \otimes 1_4) + \omega^{IJ}(1_4 \otimes \gamma_{IJ}) & 2\omega^{mI}(\gamma_m \otimes \bar{\gamma}_I) \\ 2\omega^{mI}(\bar{\gamma}_m \otimes \gamma_I) & \omega^{mn}(\bar{\gamma}_{mn} \otimes 1_4) + \omega^{IJ}(1_4 \otimes \bar{\gamma}_{IJ}) \end{pmatrix} \quad (C17)$$

The direct products  $\otimes$  are applied from (left side)  $\times$  (right side) on the  $4 \times 4$  matrices  $\psi, \psi^c$ . In this notation the  $\overline{32} \times 32$  gamma matrices  $(\bar{\Gamma}_M)^{\dot{A}B} = (\bar{\Gamma}_m, \bar{\Gamma}_I)$  act such as to mix the two spinors  $\overline{32}$  and  $32$  of  $SO(10, 2)$ , as  $\bar{\Gamma}_m(32) = (\overline{32})$ , where

$$\overline{32} = \begin{pmatrix} (4, 4) \\ (\bar{4}, \bar{4}) \end{pmatrix}, \text{ versus } 32 = \begin{pmatrix} (4, \bar{4}) \\ (\bar{4}, 4) \end{pmatrix} = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix} \quad (C18)$$

Therefore,  $(\bar{\Gamma}_M)^{\dot{A}B} = (\bar{\Gamma}_m, \bar{\Gamma}_I)^{\dot{A}B}$  must act on  $\psi_B$  as follows

$$\bar{\Gamma}_m \begin{pmatrix} (\psi)_\rho^r \\ (\psi^c)_{\dot{\rho}r} \end{pmatrix} = \begin{pmatrix} (\gamma_m \psi^c)_{\rho r} \\ (\bar{\gamma}_m \psi)_{\dot{\rho}}^r \end{pmatrix} \sim \overline{32}, \text{ and } \bar{\Gamma}_I \begin{pmatrix} (\psi)_\rho^r \\ (\psi^c)_{\dot{\rho}r} \end{pmatrix} = \begin{pmatrix} (\psi \gamma_I)_{\rho r} \\ -(\psi^c \bar{\gamma}_I)_{\dot{\rho}}^r \end{pmatrix} \sim \overline{32} \quad (C19)$$

Now we can introduce the  $SO(10, 2)$  gamma matrices  $\Gamma_M$  and  $\bar{\Gamma}_M$  as follows

$$\overline{32} \times 32, (\bar{\Gamma}_M)^{\dot{A}B} : \bar{\Gamma}_m = \begin{pmatrix} 0 & \gamma_m \otimes 1_4 \\ \bar{\gamma}_m \otimes 1_4 & 0 \end{pmatrix}, \bar{\Gamma}_I = \begin{pmatrix} 1_4 \otimes \gamma_I & 0 \\ 0 & -1_4 \otimes \bar{\gamma}_I \end{pmatrix} \quad (C20)$$

$$32 \times \overline{32}, (\Gamma_M)_{\dot{A}B} : \Gamma_m = \begin{pmatrix} 0 & \gamma_m \otimes 1_4 \\ \bar{\gamma}_m \otimes 1_4 & 0 \end{pmatrix}, \Gamma_I = \begin{pmatrix} 1_4 \otimes \bar{\gamma}_I & 0 \\ 0 & -1_4 \otimes \gamma_I \end{pmatrix} \quad (C21)$$

These  $SO(10, 2)$  gamma matrices  $\Gamma_M = (\Gamma_m, \Gamma_I)$ , and  $\bar{\Gamma}_M = (\bar{\Gamma}_m, \bar{\Gamma}_I)$  satisfy the Clifford algebra property

$$\Gamma_M \bar{\Gamma}_N + \Gamma_N \bar{\Gamma}_M = 2\eta_{MN}, \text{ and } \bar{\Gamma}_M \Gamma_N + \bar{\Gamma}_N \Gamma_M = 2\eta_{MN} \quad (C22)$$

In more detail, this is seen as follows

$$\Gamma_m \bar{\Gamma}_n + \Gamma_n \bar{\Gamma}_m = \begin{pmatrix} 0 & \gamma_m \otimes 1_4 \\ \bar{\gamma}_m \otimes 1_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma_n \otimes 1_4 \\ \bar{\gamma}_n \otimes 1_4 & 0 \end{pmatrix} + (m \leftrightarrow n) \quad (C23)$$

$$= \begin{pmatrix} (\gamma_m \bar{\gamma}_n + (m \leftrightarrow n)) \otimes 1_4 & 0 \\ 0 & (\bar{\gamma}_m \gamma_n + (m \leftrightarrow n)) \otimes 1_4 \end{pmatrix} = 2\eta_{mn} \quad (C24)$$

Similarly,

$$\Gamma_I \bar{\Gamma}_J + \Gamma_J \bar{\Gamma}_I = \begin{pmatrix} 1_4 \otimes \bar{\gamma}_I & 0 \\ 0 & -1_4 \otimes \gamma_I \end{pmatrix} \begin{pmatrix} 1_4 \otimes \gamma_J & 0 \\ 0 & -1_4 \otimes \bar{\gamma}_J \end{pmatrix} + (I \leftrightarrow J) \quad (C25)$$

$$= \begin{pmatrix} 1_4 \otimes (\gamma_J \bar{\gamma}_I + (I \leftrightarrow J)) & 0 \\ 0 & 1_4 \otimes (\bar{\gamma}_J \gamma_I + (I \leftrightarrow J)) \end{pmatrix} = 2\delta_{IJ} \quad (C26)$$

Note that in computing the products above the orders in the second factor are reversed because the second factor in the direct product is applied from the *right side* as emphasized above. Finally,

$$\Gamma_m \bar{\Gamma}_I + \Gamma_I \bar{\Gamma}_m = \begin{pmatrix} 0 & \gamma_m \otimes 1_4 \\ \bar{\gamma}_m \otimes 1_4 & 0 \end{pmatrix} \begin{pmatrix} 1_4 \otimes \gamma_I & 0 \\ 0 & -1_4 \otimes \bar{\gamma}_I \end{pmatrix} \quad (\text{C27})$$

$$+ \begin{pmatrix} 1_4 \otimes \bar{\gamma}_I & 0 \\ 0 & -1_4 \otimes \gamma_I \end{pmatrix} \begin{pmatrix} 0 & \gamma_m \otimes 1_4 \\ \bar{\gamma}_m \otimes 1_4 & 0 \end{pmatrix} \quad (\text{C28})$$

$$= \begin{pmatrix} 0 & -(\gamma_m \otimes \bar{\gamma}_I) + (\gamma_m \otimes \bar{\gamma}_I) \\ (\bar{\gamma}_m \otimes \gamma_I) - (\bar{\gamma}_m \otimes \gamma_I) & 0 \end{pmatrix} \quad (\text{C29})$$

$$= 0 \quad (\text{C30})$$

In the same way we compute  $(\Gamma_{MN})_A^B = \frac{1}{2} (\Gamma_M \bar{\Gamma}_N - \Gamma_N \bar{\Gamma}_M)_A^B$  and find

$$\Gamma_{mn} = \begin{pmatrix} \gamma_{mn} \otimes 1_4 & 0 \\ 0 & \bar{\gamma}_{mn} \otimes 1_4 \end{pmatrix}, \quad \Gamma_{IJ} = \begin{pmatrix} -1_4 \otimes \gamma_{IJ} & 0 \\ 0 & -1_4 \otimes \bar{\gamma}_{IJ} \end{pmatrix}, \quad (\text{C31})$$

$$\Gamma_{mI} = \begin{pmatrix} 0 & -\gamma_m \otimes \bar{\gamma}_I \\ \bar{\gamma}_m \otimes \gamma_I & 0 \end{pmatrix} \quad (\text{C32})$$

The matrices  $\frac{1}{2i} \gamma_{mn}$  close under commutation to form the 32×32 spinor representation of the SO(10, 2) Lie algebra.

There is an antisymmetric SO(10, 2) invariant tensor  $a_{AB} = -a_{BA}$  in the space of the spinors since  $(32 \times 32)_{\text{antisymm}}$  contains the SO(10, 2) singlet, namely the matrix  $a$  satisfies  $\delta_\omega a = 0$ , or  $\Gamma_{MN} a + a (\Gamma_{MN})^T = 0$ . Taking the antisymmetry of  $a$  into account, this implies that the matrices  $(\gamma_{mn} a)_{AB}$  are symmetric under the interchange of  $A \leftrightarrow B$ ,  $(\Gamma_{MN} a) = (\Gamma_{MN} a)^T$ . The explicit matrix  $a$  is given by

$$a_{AB} = \begin{pmatrix} 0 & C \otimes 1_4 \\ \bar{C} \otimes 1_4 & 0 \end{pmatrix}. \quad (\text{C33})$$

Recalling the following symmetry properties of the gamma matrices under transposition (appendix in [2])

$$(\Gamma^M \bar{C})^T = -(\Gamma^M \bar{C}), \quad (\Gamma^{MN} C)^T = (\bar{\Gamma}^{MN} C), \quad (\text{C34})$$

$$(\Gamma^I)^T = -(\Gamma^I), \quad (\Gamma^{IJ})^T = (\bar{\Gamma}^{IJ}), \quad (\text{C35})$$



we verify explicitly that indeed  $(\Gamma_{MN}a) = (\Gamma_{MN}a)^T$  is satisfied. The matrix  $a_{AB}$ , together with its inverse  $a^{AB}$ , plays the role of an invariant metric that can be used to raise or lower indices in the 32-spinor space.

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